An extension of the Error Correcting Pairs algorithm

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Inria, LIX

GT BAC
18/04/2018

Former Decoding Algorithms for Reed-Solomon codes

Error Correcting Pairs algorithm

PECP for Reed-Solomon codes

PECP for Algebraic Geometry codes

Former Decoding Algorithms for Reed-Solomon codes

Reed-Solomon codes

Let $x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$ such that $x_i \neq x_j$ for all $i \neq j$. Given $k \in \mathbb{N}$ such that $k \leq n$,

$$RS[n,k](x) := \{(f(x_1), \ldots, f(x_n)) \mid f \in \mathbb{F}_q[X]_{<k}\}.$$

$RS[n,k]$ is a linear code of **length** n and **dimension** k.
Reed-Solomon codes are **MDS**, that is $d = n - k + 1$.

Notation: $\mathrm{ev}_x(f) = (f(x_1), \ldots, f(x_n))$.

Problem

Let $C = RS[n, k] \subseteq \mathbb{F}_q^n$ and $y \in \mathbb{F}_q^n$. Given $t \in \mathbb{N}$, find a codeword c such that

$$\mathrm{d}(y, c) \leq t.$$

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Hypothesis

There exist $c = (\mathrm{ev}_x(f)) \in C$ with $\deg(f) < k$ and $e = (e_1, \ldots, e_n) \in \mathbb{F}_q^n$ with $\mathrm{w}(e) = t$ such that

$$y = c + e.$$

We denote the support of the error vector by

$$I = \{i \in \{1, \ldots, n\} \mid e_i \neq 0\}.$$

Algorithms for Reed Solomon codes

$t \leq \left\lfloor \frac{d-1}{2} \right\rfloor$ Berlekamp-Welch [1]

[1] L. R. Welch, E.R.Berlekamp. Error Correction for Algebraic Block Codes. United States Patent, 1986.

Key Equations (Roth)

Let $\Lambda(X) \coloneqq \prod_{i\in I}(X - x_i)$. Then, for all $i = 1, \ldots, n$ it holds

$$\Lambda(x_i)y_i = \Lambda(x_i)f(x_i).$$

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Linearisation

BW Problem: find (λ, γ) with $\deg(\lambda) \leq t$ and $\deg(\gamma) \leq t + k - 1$ such that

$$\lambda(x_i) y_i = \gamma(x_i) \quad \forall i = 1, \ldots, n.$$

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$$\lambda(x_i) y_i = \gamma(x_i) \quad \forall i = 1, \ldots, n.$$

Theorem

If $t \leq \frac{d-1}{2}$, then for all solutions (λ, γ) to BW Problem with $\lambda \neq 0$, it holds $\frac{\gamma}{\lambda} = f$.

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$t > \left\lfloor \frac{d-1}{2} \right\rfloor$ Sudan [2]

[2] M. Sudan. Decoding of Reed-Solomon codes beyond the error-correction bound. Journal of Complexity, 1997.

Let us consider the key equations of Berlekamp-Welch algorithm

$$\Lambda(x_i)f(x_i) - \Lambda(x_i)y_i = 0 \quad \forall i = 1, \ldots, n.$$

New formulation of BW Problem for $t = \frac{n-k}{2}$

Look for a polynomial $Q(X, Y) = Q_0(X) + Q_1(X)Y$ such that

- $Q(x_i, y_i) = 0$ for all $i = 1, \ldots, n$;
- $\deg(Q_j) < n - t - j(k-1)$ for $j = 0, 1$.

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Berlekamp-Welch algorithm (new formulation)

- find $Q(X, Y) \neq 0$ as above;
- return $f = -\frac{Q_0}{Q_1}$.

Sudan algorithm $\ell \geq 1$

Interpolation problem

Find $Q(X, Y) = Q_0(X) + \cdots + Q_\ell(X)Y^\ell \in \mathbb{F}_q[X, Y]$ such that

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Theorem

Le $Q(X,Y) \neq 0$ be as above. If $f(X)$ is such that $\deg(f) < k$ and $\mathrm{d}(\mathrm{ev}_x(f), y) \leq t$, then $(Y - f(X)) | Q(X,Y)$.

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Sudan algorithm

- find $Q(X, Y) \neq 0$ as above;
- find the factors of $Q(X, Y)$ linear in Y.

Remark

$$\exists Q(X, Y) \neq 0 \text{ as above} \implies \text{Sudan algorithm works}$$

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$$\#equations < \#unknowns.$$

That gives for a general ℓ the **decoding radius**

$$t \leq \frac{2n\ell - k\ell(\ell+1) + \ell(\ell+1) - 2}{2(\ell+1)}.$$

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Complexity

The most expensive step is the interpolation that gives $O(n^3\ell)$.

Algorithms for Reed Solomon codes

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$$\downarrow$$

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Power Decoding [3]

[3] G. Schmidt, V. R. Sidorenko, M. Bossert. Syndrome Decoding of Reed-Solomon Codes Beyond Half of the Minimum Distance based on Shift-Register Synthesis. IEEE Transactions on Information Theory, 2010.

Star (Schur) Product

Given $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ in $\mathbb{F}^n$

- $a * b = (a_1 b_1, \ldots, a_n b_n)$;
- $a^{*2} = a * a$.

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Given $A, B \subseteq \mathbb{F}^n$

- $A * B = \mathrm{span}_{\mathbb{F}}(\{a * b \mid a \in A, b \in B\})$;
- $A^{*2} = A * A$.

Power Decoding algorithm $\ell = 2$

Let us define e' this way

$$y^{*2} = c^{*2} + \underbrace{2c * e + e^{*2}}_{e'}.$$

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Lemma

We get $\mathrm{supp}(e') \subseteq I = \mathrm{supp}(e)$.

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Key Equations (Rosenkilde)

Let $\Lambda(X) \coloneqq \prod_{i \in I}(X - x_i)$. Then, for all $i = 1, \ldots, n$ it holds

$$\begin{cases} \Lambda(x_i) y_i = \Lambda(x_i) f(x_i) \\ \Lambda(x_i) y_i^2 = \Lambda(x_i) f^2(x_i) \end{cases}$$

Linearisation

PwDc Problem: find $(\lambda, \gamma_1, \gamma_2)$ with $\deg(\lambda) \leq t$ and $\deg(\gamma_i) \leq t + i(k-1)$ for $i = 1, 2$ such that

$$\begin{cases} \lambda(x_i)y_i = \gamma_1(x_i) & \forall i = 1, \ldots, n \\ \lambda(x_i)y_i^2 = \gamma_2(x_i) & \forall i = 1, \ldots, n. \end{cases}$$

Power Decoding algorithm

- find the solution space $\mathcal{S}$ to PwDc Problem;
- pick $(\lambda, \gamma_1, \gamma_2)$ in $\mathcal{S}$ with $\lambda \neq 0$ with the minimum degree;
- if $\lambda | \gamma_1$, return $\frac{\gamma_1}{\lambda}$.

Remark

$(\Lambda, \Lambda f, \Lambda f^2)$ is a solution for PwDc Problem.

A **necessary condition** to have a solution space of dimension smaller than two, is to have

$$\#unknowns \leq \#equations + 1.$$

That gives for a general ℓ the **decoding radius**

$$t \leq \frac{2n\ell - k\ell(\ell+1) + \ell(\ell-1)}{2(\ell+1)}.$$

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Complexity

The cost of the algorithm is the one to solve a linear system of $n\ell$ equations in $O(n\ell)$ unknowns, that is $O(n^3\ell^3)$.

Algorithms for Reed Solomon codes

$t \leq \left\lfloor \frac{d-1}{2} \right\rfloor$ Berlekamp-Welch

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Power Decoding

Algorithms for Reed Solomon codes

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[4] R. Pellikaan. On decoding by error location and dependent sets of error positions. Discrete Mathematics, 1992.

Error Correcting Pairs algorithm

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Error Correcting Pairs (ECP)

Given a linear code $C \subseteq \mathbb{F}_q^n$, a couple of linear codes (A, B) with $A, B \subseteq \mathbb{F}_q^n$ is a t-error correcting pair for C if

- $A * B \subseteq C^{\perp}$;
- $\dim(A) > t$;
- $\mathrm{d}(B^{\perp}) > t$;
- $\mathrm{d}(A) + \mathrm{d}(C) > n$.

Theorem (R. Pellikaan, 1992)

Let $C \subseteq \mathbb{F}_q^n$ be a linear code. If there exists a t-error correcting pair for C, then for all $y \in \mathbb{F}_q^n$ such that

$$y = c + e,$$

with $c \in C$ and $\mathrm{w}(e) \leq t$, the ECP algorithm recovers c with complexity $O(n^3)$.

Theorem (R. Pellikaan, 1992)

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Proposition

If a linear code C has a t-error correcting pair, then

$$t \leq \left\lfloor \frac{\mathrm{d}(C) - 1}{2} \right\rfloor.$$

Given $J = \{j_1, \ldots, j_s\} \subset \{1, \ldots, n\}$ and $x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$

- $x_J \coloneqq (x_{j_1}, \ldots, x_{j_s})$ (puncturing);
- $Z(x) \coloneqq \{i \in \{1, \ldots, n\} \mid x_i = 0\}$.

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Moreover, if $A \subseteq \mathbb{F}_q^n$

- $A_J := \{a_J \mid a \in A\} \subseteq \mathbb{F}_q^{|J|}$;
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- $Z(A) := \{i \in \{1, \dots, n\} \mid a_i = 0 \;\; \forall a \in A\}$;
- $A(J) := \{a \in A \mid a_J = 0\} \subseteq \mathbb{F}_q^n$ (shortening).

We define $M := \{a \in A \mid \langle a * y, b\rangle = 0 \;\; \forall b \in B\}$.

Lemma

Let y, $I = \mathrm{supp}(e)$ and M as above. If $A * B \subseteq C^{\perp}$, then

- $A(I) \subseteq M \subseteq A$;

Localisation of errors

We define $M \coloneqq \{a \in A \mid \langle a * y, b \rangle = 0 \;\; \forall b \in B\}$.

Lemma

Let y, $I = \mathrm{supp}(e)$ and M as above. If $A * B \subseteq C^{\perp}$, then

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Proof of $A(I) \subseteq M$: given $a \in A(I)$, we get for all $b \in B$

$$\langle a * y, b\rangle = \underbrace{\langle a * c, b\rangle}_{\langle a*b,c\rangle} + \underbrace{\langle a * e, b\rangle}_{\langle 0,b\rangle} = 0.$$

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$\longrightarrow$ we take $J := Z(M)$.

Let $H \in \mathcal{M}(n, m)$, and H^i its columns. Given $J \subseteq \{1, \ldots, m\}$, we define

$$H_J = (H^j)^{j \in J}.$$

Let us consider a full rank parity check matrix H for C.

Lemma

If $\mathrm{d}(A) + \mathrm{d}(C) > n$ and $I \subseteq J$, then there exists an unique solution for the system

$$H_J \cdot E^T = H \cdot y^T.$$

Recovering e

Let $H \in \mathcal{M}(n, m)$, and H^i its columns. Given $J \subseteq \{1, \dots, m\}$, we define

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$\longrightarrow$ we recover e.

PECP for Reed-Solomon codes

Let $C \subseteq \mathbb{F}_q^n$ be a RS[n,k] code. There exists $f \in \mathbb{F}_q[x]_{<k}$ such that $c = (\mathrm{ev}_x(f))$. Let us take

$$A = RS[n, t+1], \qquad B^{\perp} = RS[n, t+k].$$

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$\dim(A) > t$ obvious

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Proposition

We have that $\mathrm{d}(B^{\perp}) > t$ if and only if

$$t \leq \left\lfloor \frac{\mathrm{d}(C) - 1}{2} \right\rfloor.$$

Berlekamp-Welch key equations and the choice of M

Berlekamp Welch algorithm's key equation

Given $\Lambda(X) = \prod_{i \in I}(X - x_i)$ and $N(X) \coloneqq \Lambda(X) f(X)$, it holds

$$\mathrm{ev}_x(\Lambda) * y = \mathrm{ev}_x(N).$$

We get

- $(N(x_1), \ldots, N(x_n)) \in B^{\perp} = RS[t + k]$;
- $(\Lambda(x_1), \ldots, \Lambda(x_n)) \in A(I) = RS[t + 1](I)$;

Berlekamp-Welch key equations and the choice of M

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We get

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- $(\Lambda(x_1), \ldots, \Lambda(x_n)) \in A(I) = RS[t + 1](I)$;
- $(\Lambda(x_1), \ldots, \Lambda(x_n)) \in \underbrace{\{a \in A \mid \langle a * y, b \rangle = 0 \ \ \forall b \in B\}}_{M}$.

Algorithms for Reed Solomon codes

$t \leq \left\lfloor \frac{d-1}{2} \right\rfloor$	Berlekamp-Welch	Error Correcting Pairs
	$\downarrow$	$\downarrow$
$t > \left\lfloor \frac{d-1}{2} \right\rfloor$	Sudan Power Decoding	?

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Proposition

We have that $\mathrm{d}(B^{\perp}) > t$ if and only if

$$t \leq \left\lfloor \frac{\mathrm{d}(C) - 1}{2} \right\rfloor.$$

Power Error Correcting Pairs algorithm with power $\ell = 2$

Error Locating Pair

Given A, B, C linear codes of length n, (A, B) is a t-error locating pair (ELP) for C if

- $A * B \subseteq C^{\perp}$;
- $\dim(A) > t$;
- $\mathrm{d}(A) + \mathrm{d}(C) > n$.

Pellikaan, 1992:

> *If I is an independent t-set of error positions with respect to B, where (A, B) is a t-error locating pair for C, then the algorithm corrects any word with error supported at I.*

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Before we used "If $A * B \subseteq C^{\perp}$ and $\mathrm{d}(B^{\perp}) > t$, then $A(I) = M$."

Let us define

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Power Decoding algorithm's key equations

Given $\Lambda(X) = \prod_{i \in I}(X - x_i)$ as before, then

$$\begin{cases} \mathrm{ev}_x(\Lambda) * y = \mathrm{ev}_x(N_1) \\ \mathrm{ev}_x(\Lambda) * y^{*2} = \mathrm{ev}_x(N_2) \end{cases} .$$

Hence, if we consider $A = RS[n, t+1]$, $B^\perp = RS[n, t+k]$ as before, we get

- $(N_1(x_1), \ldots, N_1(x_n)) \in B^\perp$;
- $(N_2(x_1), \ldots, N_2(x_n)) \in B^\perp * C$;

Hence, if we consider $A = RS[n, t+1]$, $B^\perp = RS[n, t+k]$ as before, we get

- $(N_1(x_1), \dots, N_1(x_n)) \in B^\perp$;
- $(N_2(x_1), \dots, N_2(x_n)) \in B^\perp * C$;
- $(\Lambda(x_1), \dots, \Lambda(x_n)) \in A(I),\ M_1 \cap M_2$.

where M_1 and M_2 are defined this way

$$M_1 := \{a \in A \mid \langle a * y, b \rangle = 0 \ \ \forall b \in B\},$$
$$M_2 := \{a \in A \mid \langle a * y^{*2}, v \rangle = 0 \ \ \forall v \in (B^\perp * C)^\perp\}.$$

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$\longrightarrow$ we take $M = M_1 \cap M_2$.

Lemma

If $A * B \subseteq C^{\perp}$, then $A(I) \subseteq M = M_1 \cap M_2 \subseteq A$.

PECP algorithm:

- compute $M = M_1 \cap M_2$ (linear system);
- compute $J = Z(M)$;
- solve the syndrom linear system.

This algorithm can be runned on all codes with an ELP.

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Since $M(I) = A(I)$, we get the implications:

$$M = A(I) \iff M(I) = M \iff M_I = \{0\}.$$

Given $a \in A$, we have by definition of M_1

$$a \in M_1 \iff \langle a * y, b \rangle = 0 \quad \forall b \in B.$$

If $A * B \subseteq C^\perp$, this is equivalent to $a_I \in (e * B)_I^\perp$.

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In the same way, given $a \in A$, it holds

$$a \in M_2 \iff a_I \in (e' * (B^\perp * C)^\perp)_I^\perp.$$

Lemma

We have $(M_1 \cap M_2)_I = (e * B)_I^\perp \cap (e' * (B^\perp * C)^\perp)_I^\perp \cap A_I$.

Remark

Since $A = RS[n, t+1]$ is MDS, then $A_I = \mathbb{F}_q^t$.

Hence $(M_1 \cap M_2)_I = (e * B)_I^\perp \cap (e' * (B^\perp * C)^\perp)_I^\perp$.

Remark

Since $A = RS[n, t+1]$ is MDS, then $A_I = \mathbb{F}_q^t$.

Hence $(M_1 \cap M_2)_I = (e * B)_I^\perp \cap (e' * (B^\perp * C)^\perp)_I^\perp$.

A necessary condition for $(M_1 \cap M_2)_I$ to be the null space is

$$\dim((e * B)_I^\perp) + \dim((e' * (B^\perp * C)^\perp)_I^\perp) \leq t.$$

This inequality implies the following

Necessary condition

$$\dim(B) + \dim((B^\perp * C)^\perp) \geq t.$$

Decoding radius for Reed-Solomon codes and $\ell = 2$

We get, as for the Power Decoding algorithm with power 2,

$$t \leq \frac{2n - 3k + 1}{3}.$$

It is possible to write the algorithm for a general power ℓ.

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For **Reed-Solomon codes**, PECP has the same decoding radius as the Power Decoding algorithm, that is $t_{pow} = \frac{2n\ell - k\ell(\ell+1) + \ell(\ell-1)}{2(\ell+1)}$.

Complexity

PECP(ℓ):

(i) find $M = \bigcap_{i=1}^{\ell} M_i$;

(ii) given J, find c.

The main cost is the one of step (i), which reduces to a linear system of $O(n\ell)$ equations in

$$t + 1 = O\left(\frac{2n\ell + \ell(\ell+1) + 2}{2(\ell+1)}\right) = O(n)$$

unknowns. Hence we get the cost $O(n^3\ell)$, while the cost of Power Decoding algorithm is $O(n^3\ell^3)$.

PECP for Algebraic Geometry codes

Let χ be a smooth projective curve, $\mathcal{P} = \{P_1, \ldots, P_n\} \subseteq \chi$, G a divisor for χ with $\text{supp}(G) \cap \mathcal{P} = \emptyset$ and

$$C = C_L(\chi, \mathcal{P}, G).$$

Theorem

There exists a t-error locating pair for C such that the necessary condition gives the correcting radius

$$t \leq \underbrace{\frac{2n\ell - \ell(\ell+1)\deg(G) - 2\ell}{2(\ell+1)} - g}_{t_{basic}, t_{pow}[SW98]} + \frac{g}{\ell+1}.$$

As for Reed-Solomon codes, the PECP algorithm costs $O(n^3\ell)$, while the Power Decoding algorithm costs $O(n^3\ell^3)$.

Future tasks:

- study of the failure cases of the Power Decoding algorithm and the PECP algorithm for Reed-Solomon codes;
- examine the possibility to improve PECP algorithm's decoding radius for algebraic-geometry codes;
- is it possible to design a multiplicity version of ECP algorithm?

Thanks for your attention!