Faster fully homomorphic encryption: Bootstrapping in less than 0.1 seconds

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April 6, 2017

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Homomorphic Encryption

IDEA: perform computations on encrypted data, without decrypting it.

$b_1, b_2 \in \{0, 1\}$

$$\boxed{b_1} \qquad\qquad \boxed{b_1} \oplus_{hom} \boxed{b_2} \quad = \quad \boxed{b_1 \oplus b_2}$$

$$\longrightarrow$$

$$\boxed{b_2} \qquad\qquad \boxed{b_1} \wedge_{hom} \boxed{b_2} \quad = \quad \boxed{b_1 \wedge b_2}$$

Homomorphic Encryption

More generally

$$\boxed{b_1} \\ \vdots \\ \boxed{b_n} \quad \longrightarrow \quad \varphi_{hom}(\boxed{b_1}, \ldots, \boxed{b_n}) \quad = \quad \boxed{\varphi(b_1, \ldots, b_n)}$$

where $b_1, \ldots, b_n \in \{0, 1\}$ and φ is a boolean circuit.

Homomorphic Encryption

An **Homomorphic Encryption** scheme is composed by 4 algorithms:

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- **Key Generation** KeyGen :

$$\lambda \longmapsto (sk, pk)$$

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$$(c, sk) \longmapsto m$$

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- **Key Generation** KeyGen :

$$\lambda \longmapsto (sk, pk)$$

- **Decryption** Dec (deterministic) :

$$(c, sk) \longmapsto m$$

- **Encryption** Enc (randomized) :

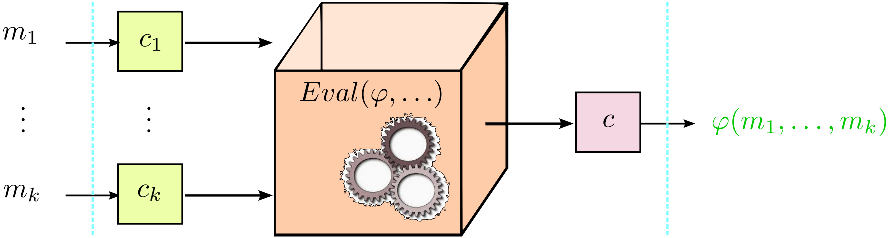
$$(m, pk) \longmapsto c$$

  such that $\mathsf{Dec}(c, sk) = m$

- **Evaluation** Eval (possibly randomized) :

$$(\varphi, c_1, \ldots, c_k) \longmapsto c$$

such that $\mathsf{Dec}(c, sk) = \varphi(m_1, \ldots, m_k)$

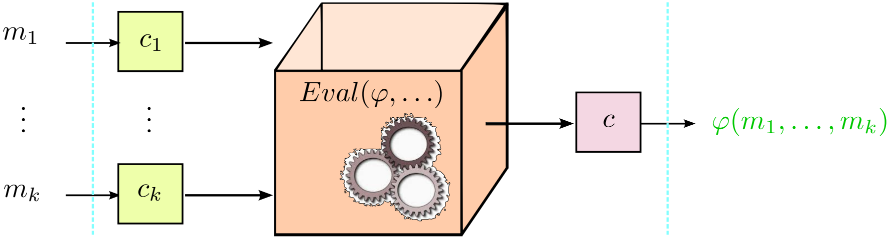


Homomorphic Encryption

- **Evaluation** Eval (possibly randomized) :

$$(\varphi, c_1, \ldots, c_k) \longmapsto c$$

such that $\mathsf{Dec}(c, sk) = \varphi(m_1, \ldots, m_k)$



A scheme that can homomorphically evaluate all functions/circuits is said **Fully Homomorphic** (FHE).

Applications

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Statistic computations on sensitive data

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Secure multiparty computation

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Electronic voting

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Cloud computing

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Statistic computations on sensitive data

Secure multiparty computation

Electronic voting

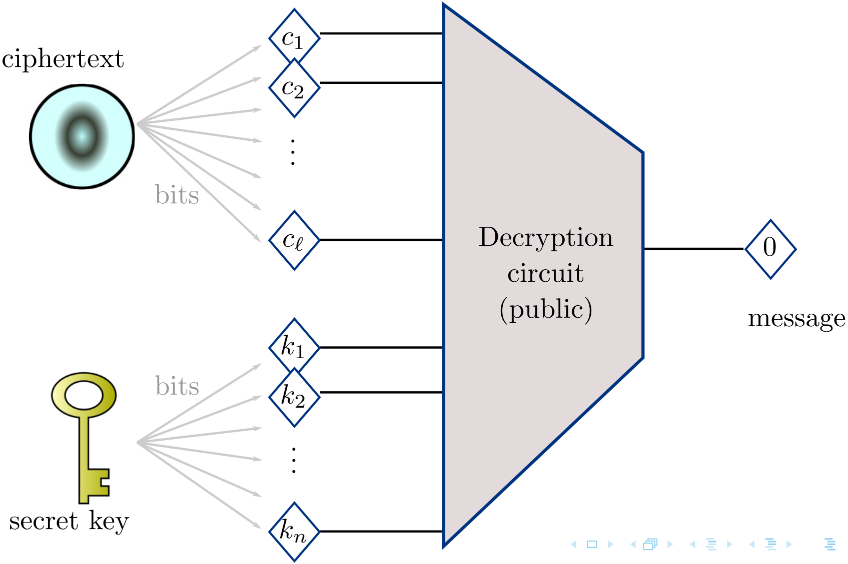
Cloud computing

and even more...

A world full of noise...

anim.html

Bootstrapping now



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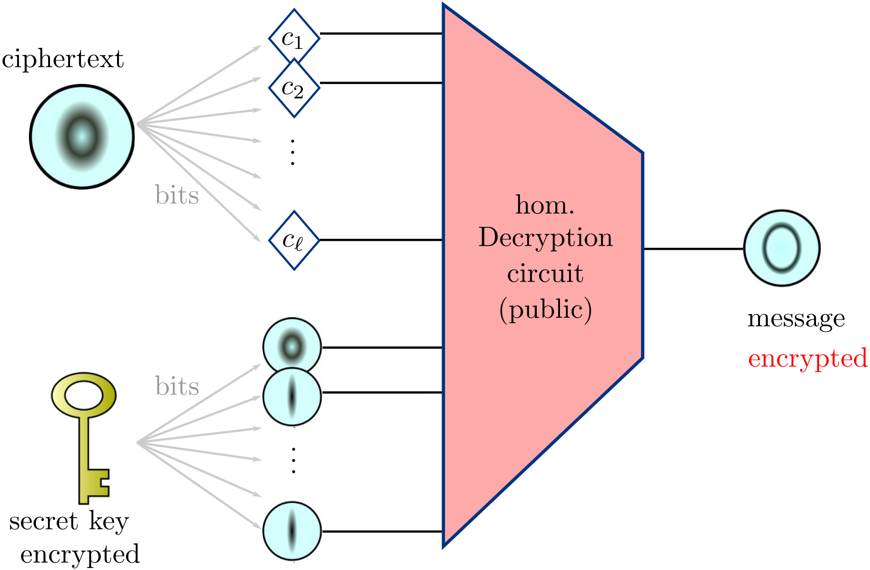

Bootstrapping now

Bootstrapping is the most expensive part of the entire homomorphic procedure

- Original idea by Gentry [Gen09]
- Last years: work to reduce the execution time and memory consuming

...but a lot have to be done!

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LWE

- LWE = Learning With Errors [Reg05]
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LWE

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In our paper

- LWE: definition similar to [BLPRS13],[CS15],[CGGI16]
- TLWE: generalized definition similar to [BGV12]

The real torus $\mathbb{T} = \mathbb{R}/\mathbb{Z} = \mathbb{R} \mod 1$

$(\mathbb{T}, +, \cdot)$ is a $\mathbb{Z}$-module ($\cdot : \mathbb{Z} \times \mathbb{T} \to \mathbb{T}$ a valid external product)

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Vectors/matrices

By extension, $(\mathbb{T}^n, +, .)$ is a $\mathbb{Z}$-module

- $\begin{bmatrix} 3 & -2 & 4 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & -2 \\ 3 & 4 \\ 5 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.252 & 0.672 \\ 0.231 & 0.991 \end{bmatrix} \right)$
- $= \left(\begin{bmatrix} 3 & -2 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & -2 \\ 3 & 4 \\ 5 & 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 0.252 & 0.672 \\ 0.231 & 0.991 \end{bmatrix}$

Torus polynomials $\mathbb{T}_N[X]$

$(\mathbb{T}_N[X], +, \cdot)$ is a $\mathfrak{R}$-module

- Here, $\mathfrak{R} = \mathbb{Z}[X]/(X^N + 1)$
- And $\mathbb{T}_N[X] = \mathbb{T}[X] \mod (X^N + 1)$

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- $(1 + 2X) \cdot (\frac{1}{3} + \frac{4}{7}X) =$

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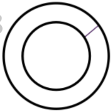
- Decompose $(\frac{3}{8} + \frac{7}{8}X)$ over $[\frac{1}{2}, \frac{1}{4}, \frac{1}{8}]$ with **small** coefs
  $(\frac{3}{8} + \frac{7}{8}X) = (0 + X) \cdot \frac{1}{2} + (1 + X) \cdot \frac{1}{4} + (1 + X) \cdot \frac{1}{8}$

LWE symmetric encryption

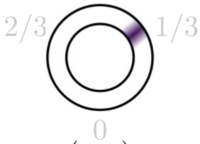
LWE symmetric encryption



Example: $\mathcal{M} = \{0, 1/3, 2/3\} \; mod \; 1$

$\mu = 1/3 \; mod \; 1 \in \mathcal{M}$

LWE symmetric encryption



$(\ , \varphi)$

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LWE Encryption

1. Choose $\varphi = \mu + \mathit{Gaussian\ Error}$

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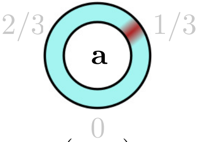


$(\mathbf{a}, \varphi)$

Example: $\mathcal{M} = \{0, 1/3, 2/3\}$ mod 1

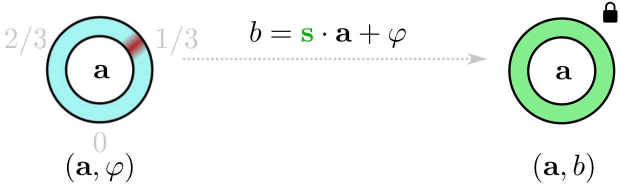
$\mu = 1/3$ mod $1 \in \mathcal{M}$

LWE Encryption

1. Choose $\varphi = \mu + \textit{Gaussian Error}$
2. Choose a random mask $\mathbf{a} \in \mathbb{T}^n$

LWE symmetric encryption

secret key: $\mathbf{s} \in \{0,1\}^n$



Example: $\mathcal{M} = \{0, 1/3, 2/3\}$ *mod* 1

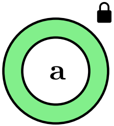
$\mu = 1/3$ *mod* $1 \in \mathcal{M}$

LWE Encryption

1. Choose $\varphi = \mu + $ *Gaussian Error*
2. Choose a random mask $\mathbf{a} \in \mathbb{T}^n$
3. Return the locked representation $(\mathbf{a}, b)$

LWE symmetric encryption

secret key: $\mathbf{s} \in \{0,1\}^n$

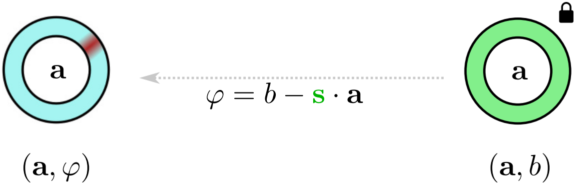


$(\mathbf{a}, b)$

LWE Decryption

LWE symmetric encryption

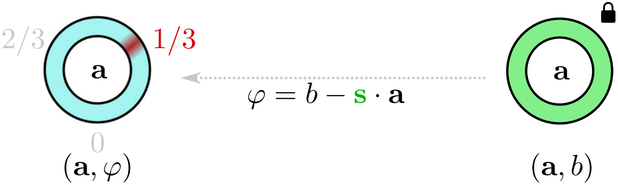
secret key: $\mathbf{s} \in \{0,1\}^n$



LWE Decryption

1. Unlock the representation $(\mathbf{a}, \varphi)$

secret key: $\mathbf{s} \in \{0,1\}^n$

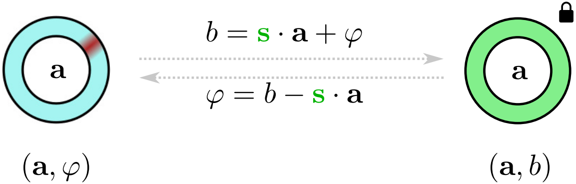


LWE Decryption

1. Unlock the representation $(\mathbf{a}, \varphi)$
2. Round φ to the nearest message $\mu \in \mathcal{M}$

LWE symmetric encryption

secret key: $\mathbf{s} \in \{0,1\}^n$



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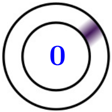
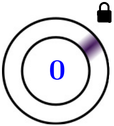


$b = \mathbf{s} \cdot \mathbf{a} + \varphi$

$\varphi = b - \mathbf{s} \cdot \mathbf{a}$



$(\mathbf{a}, b)$

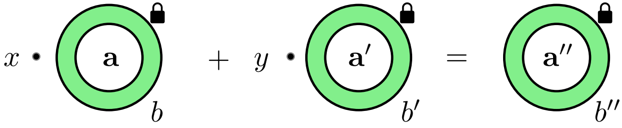
Trivial LWE samples

- LWE samples with mask $\mathbf{a} = \mathbf{0}$ are trivial.
- They never occur in general

...but are still worth mentionning!

LWE

Homomorphic Properties



$$\mathbf{a}'' = x \cdot \mathbf{a} + y \cdot \mathbf{a}'$$

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LWE

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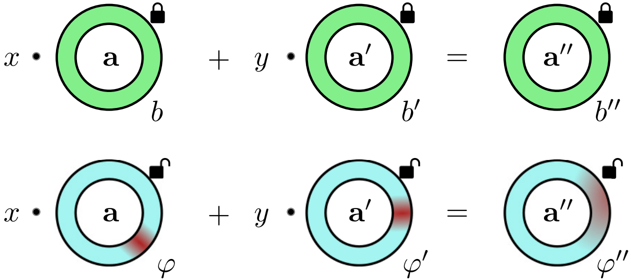

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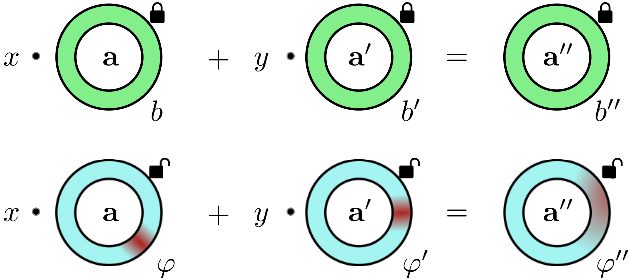

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$\mu = \mathbb{E}(\varphi)$ $\qquad \mu'$ $\qquad \mu''$ $\qquad \mu'' = x \cdot \mu + y \cdot \mu'$

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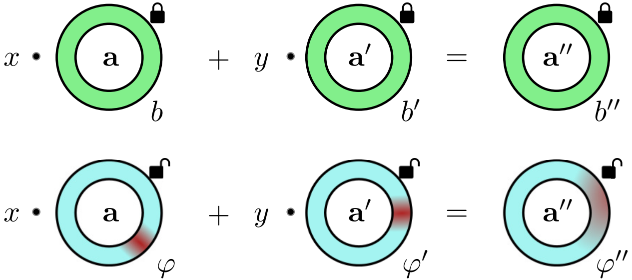

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LWE

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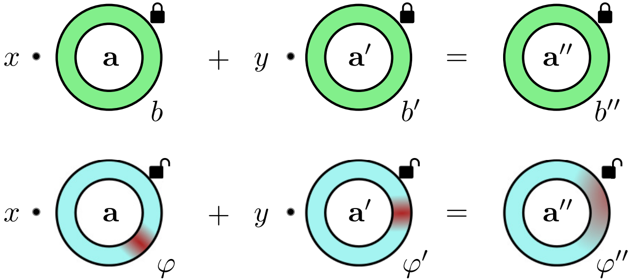

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Ω: *The only proba. space where this intuitive picture makes sense!*

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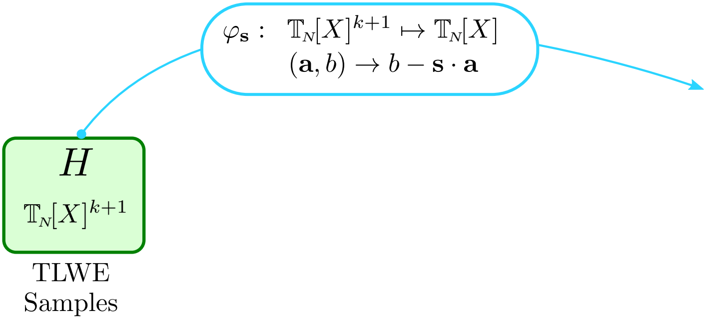

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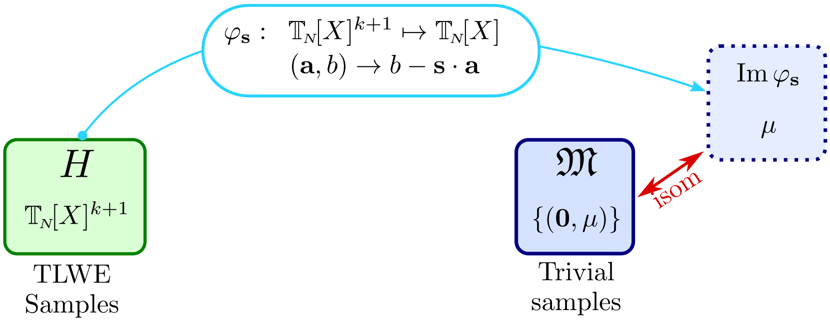

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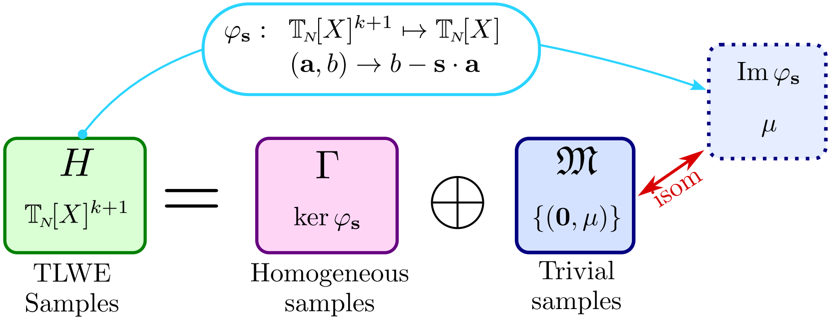

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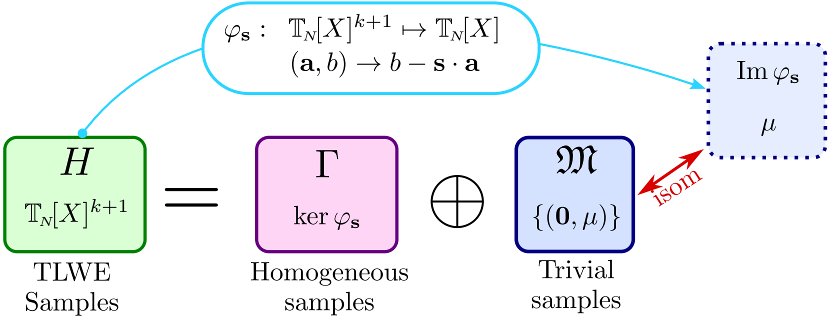

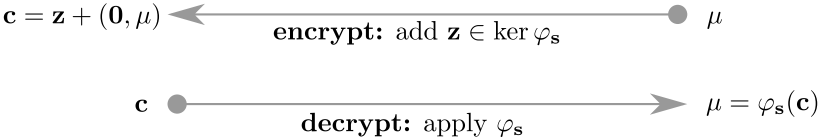

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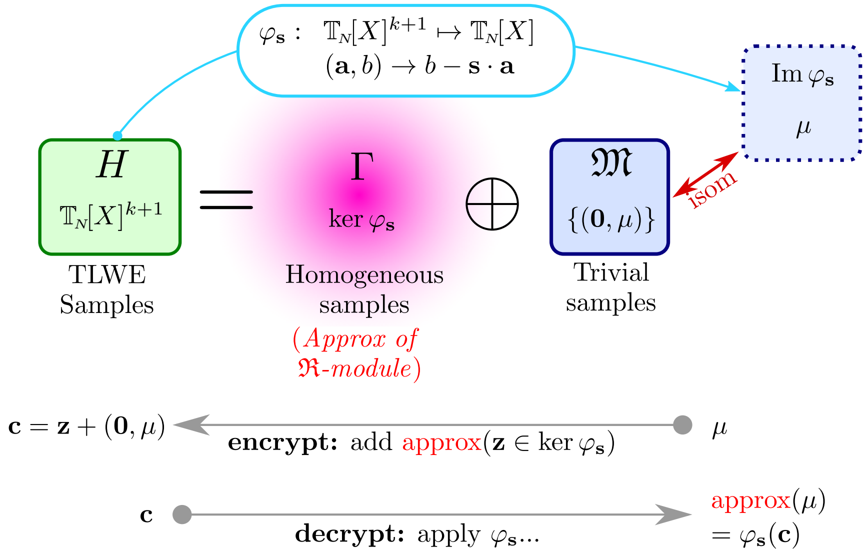

TLWE Encryption

n $\varphi_{\mathbf{s}}$

μ

How to recover μ **exactly**?

Option 1: $\mu = \mathbb{E}(\varphi_{\mathbf{s}}(\mathbf{c}))$

(in the relevant proba. space)

The Ω-space logic

Option 2: $\mu = \text{round}(\varphi_{\mathbf{s}}(\mathbf{c}))$

On a given finite message space $\mathcal{M}$

The logic of the decryption algorithm

$\mathbf{c} =$

$\mathbf{c}$ → **decrypt:** apply $\varphi_{\mathbf{s}}$...

approx(μ)

$= \varphi_{\mathbf{s}}(\mathbf{c})$

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We want FHE!

What is still missing to have **Fully** Homomorphic Encryption?

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What is still missing to have **Fully** Homomorphic Encryption?

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- Relies on a gadget decomposition function

GSW

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What is still missing to have **Fully** Homomorphic Encryption?

- GSW [GSW13] is a FHE scheme based on LWE
- Relies on a gadget decomposition function

In this talk

- Abstraction of [GSW13] by [GINX16]
- TGSW: "GSW" on $\mathbb{T}$

The gadget

$$\mathbf{v} = \left(\; v_1 \; | \ldots | \; v_{k+1} \; \right) \in H$$

$$\mathbf{h} = \left(\begin{array}{c|c|c} 1/2 & \ldots & 0 \\ 1/2^2 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 1/2^\ell & \ldots & 0 \\ \hline \vdots & \ddots & \vdots \\ \hline 0 & \ldots & 1/2 \\ 0 & \ldots & 1/2^2 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 1/2^\ell \end{array} \right)$$

h generating family of H

$\mathbf{h} \in \mathcal{M}_{\ell',k+1}(\mathbb{T}_N[X])$

- $\mathbf{h}$ is block diagonal super-increasing
- We are able to decompose elements in the sub-module H
- The coefficients in the decomposition are small
- Approximated decomposition (up to some precision parameters)
- Improve time and memory requirements for a small amount of additional noise

TGSW

Parameters

- Let $H = \mathbb{T}_N[X]^k \times \mathbb{T}_N[X]$
- $\mathbf{h} = (h_1, \ldots, h_l) \in H^{\ell'}$ a super-increasing generating family of H
- $Dec_{\mathbf{h}}$ the "small" decomposition function from $H \to \mathfrak{R}^{\ell'}$ ($\mathfrak{R} = \mathbb{Z}[X]/(X^N + 1)$) such that

$$Dec_{\mathbf{h}}(x) \cdot \mathbf{h} = x \text{ for all } x \in H$$

- $\Gamma = \ker_{\varphi_{\mathbf{s}}}$ denotes homogeneous TLWE samples

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Encryption:

$C = Z + \mu \cdot \mathbf{h}$ where $Z \in \Gamma^{\ell'}$

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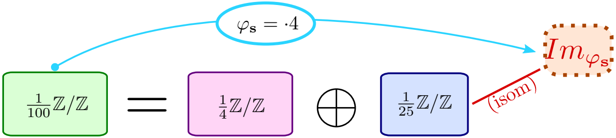
Homomorphic operations:

Let $C_1 = Z_1 + \mu_1 \cdot \mathbf{h}$ and $C_2 = Z_2 + \mu_2 \cdot \mathbf{h}$

- **Linear combinations:** $\delta_1 C_1 + \delta_2 C_2$ encrypts $\delta_1\mu_1 + \delta_2\mu_2$ ($\delta_i \in \mathfrak{R}$)
- **Multiplication :** $Dec_{\mathbf{h}}(C_1) \cdot C_2$ encrypts $\mu_1\mu_2$

Toy example (without noise)



Parameters

- $H = \frac{1}{100}\mathbb{Z}/\mathbb{Z} = \frac{1}{4}\mathbb{Z}/\mathbb{Z} \oplus \frac{1}{25}\mathbb{Z}/\mathbb{Z}$ (is a $\mathbb{Z}$-module)
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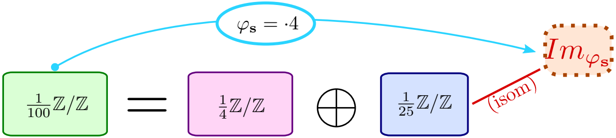

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Samples

$$C_1 = \left(\frac{32}{100}, \frac{14}{100}, \frac{60}{100}, \frac{45}{100}, \frac{90}{100}, \frac{0}{100}\right) = \left(\frac{1}{4}, \frac{0}{4}, \frac{1}{4}, \frac{3}{4}, \frac{2}{4}, \frac{2}{4}\right) + 7 \cdot \mathbf{h}$$

$$C_2 = \left(\frac{73}{100}, \frac{21}{100}, \frac{40}{100}, \frac{5}{100}, \frac{35}{100}, \frac{50}{100}\right) = \left(\frac{3}{4}, \frac{1}{4}, \frac{2}{4}, \frac{1}{4}, \frac{3}{4}, \frac{2}{4}\right) - 2 \cdot \mathbf{h}$$

Toy example (without noise)

Multiplication:

$$Dec_{\mathbf{h}}(C_1) \cdot C_2 = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 73/100 \\ 21/100 \\ 40/100 \\ 5/100 \\ 35/100 \\ 50/100 \end{bmatrix}$$

$$= \left(\frac{61}{100}, \frac{47}{100}, \frac{55}{100}, \frac{10}{100}, \frac{20}{100}, \frac{0}{100} \right)$$

Verification: does encode $7 \cdot (-2) = 11 \mod 25$

$$\left(\frac{61}{100}, \frac{47}{100}, \frac{55}{100}, \frac{10}{100}, \frac{20}{100}, \frac{0}{100} \right) = \left(\frac{2}{4}, \frac{1}{4}, \frac{0}{4}, \frac{0}{4}, \frac{0}{4}, \frac{2}{4} \right) + 11 \cdot \mathbf{h}$$

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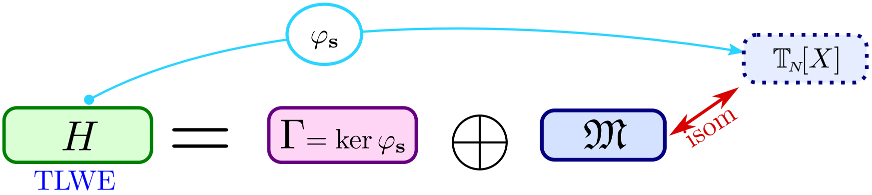
TLWE and TGSW



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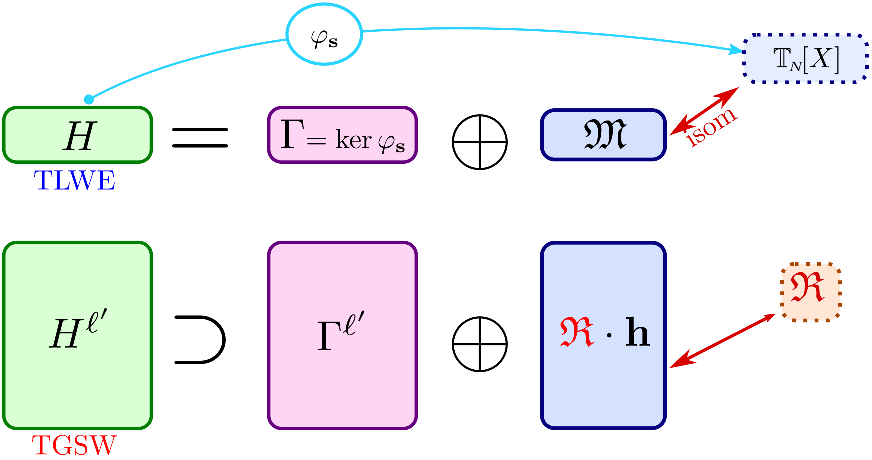

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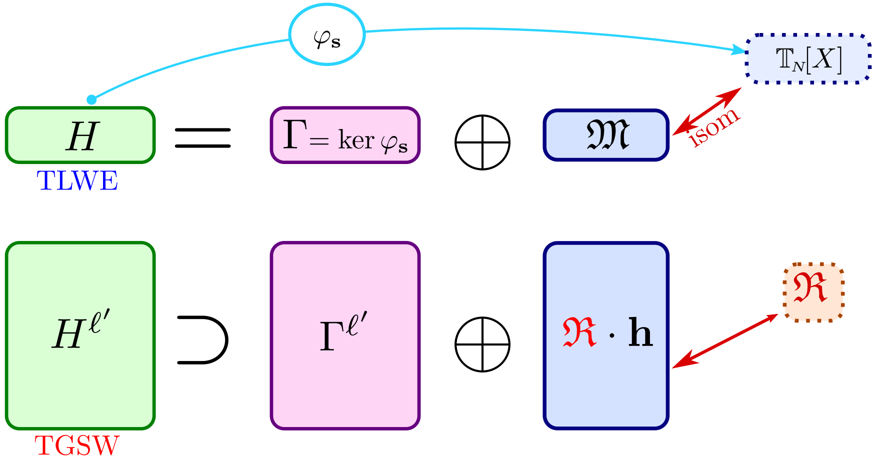
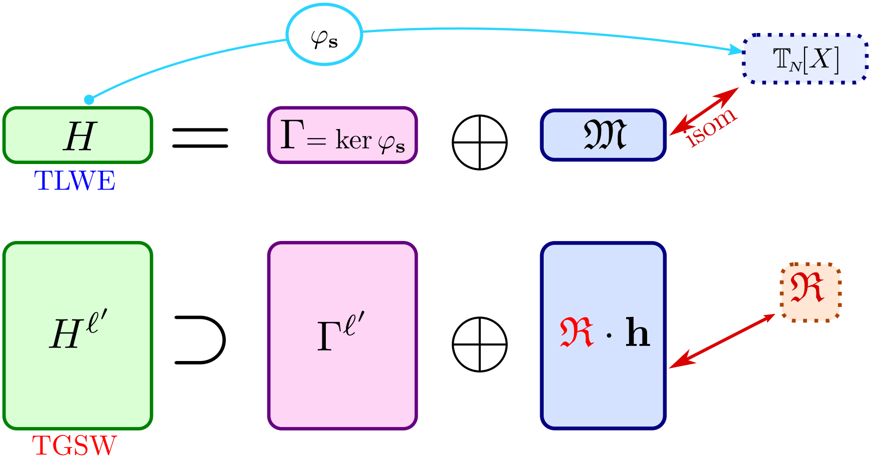


$\forall \mathbf{e} \in \mathfrak{R}^{\ell'}, \forall A \in \mathfrak{R}, \forall b \in \mathbb{T}_N[X]$:

$\mathbf{e} \cdot \mathtt{TGSW}(A)$ is a $\mathtt{TLWE}$ of $A \cdot \varphi_{\mathbf{s}}(\mathbf{e} \cdot \mathbf{h})$

TLWE and TGSW



$\forall \mathbf{e} \in \mathfrak{R}^{\ell'}, \forall A \in \mathfrak{R}, \forall b \in \mathbb{T}_N[X]$:

$\mathbf{e} \cdot \texttt{TGSW}(A)$ is a $\texttt{TLWE}$ of $A \cdot \varphi_{\mathbf{s}}(\mathbf{e} \cdot \mathbf{h})$

$\implies \mathrm{Decomp}_{\mathbf{h}}(\texttt{TLWE}(b)) \cdot \texttt{TGSW}(A)$ is a $\texttt{TLWE}$ of $A \cdot b$

Toy example (WITH noise)

Parameters

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Samples

$$C_1 = \left(\frac{31}{100}, \frac{16}{100}, \frac{63}{100}, \frac{46}{100}, \frac{89}{100}, \frac{0}{100}\right)$$

$$= \left[\left(\frac{1}{4}, \frac{0}{4}, \frac{1}{4}, \frac{3}{4}, \frac{2}{4}, \frac{2}{4}\right) + \left(-\frac{1}{100}, \frac{2}{100}, \frac{3}{100}, \frac{1}{100}, -\frac{1}{100}, \frac{1}{100}\right)\right] + 7 \cdot \mathbf{h}$$

$$C_2 = \left(\frac{71}{100}, \frac{23}{100}, \frac{37}{100}, \frac{5}{100}, \frac{33}{100}, \frac{48}{100}\right)$$

$$= \left[\left(\frac{3}{4}, \frac{1}{4}, \frac{2}{4}, \frac{1}{4}, \frac{3}{4}, \frac{2}{4}\right) + \left(-\frac{2}{100}, \frac{2}{100}, -\frac{3}{100}, \frac{0}{100}, -\frac{2}{100}, -\frac{2}{100}\right)\right] - 2 \cdot \mathbf{h}$$

Toy example (WITH noise)

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Toy example (WITH noise)

Multiplication:

$$Dec_{\mathbf{h}}(C_{1,1}) \cdot C_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 71/100 \\ 23/100 \\ 37/100 \\ 5/100 \\ 33/100 \\ 48/100 \end{bmatrix}$$

$$Dec_{\mathbf{h}}(C_{1,1}) \cdot C_2 = \left(\frac{9}{100} \right)$$

Verification: does encode $7 \cdot (-2) = 11 \mod 25$

$$\left(\frac{9}{100} \right) = \left[\left(\frac{0}{4} \right) - \left(\frac{2}{100} \right) \right] + 11 \cdot h_1$$

Product

External product (found independently by [BP16])

$$\boxdot : TGSW \times TLWE \longrightarrow TLWE$$
$$(A, \mathbf{b}) \longmapsto A \boxdot \mathbf{b} = Dec_{\mathbf{h},\beta,\epsilon}(\mathbf{b}) \cdot A$$
$$(\mu_A, \mu_{\mathbf{b}}) \longmapsto \mu_A \cdot \mu_{\mathbf{b}}$$

where $Dec_{\mathbf{h},\beta,\epsilon}$ is the approximate gadget decomposition

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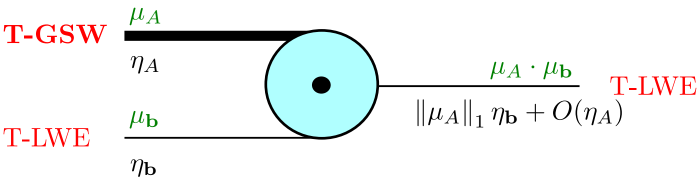
where $Dec_{\mathbf{h},\beta,\epsilon}$ is the approximate gadget decomposition

Internal product (classical)

$$\begin{aligned}\boxtimes\colon TGSW \times TGSW &\longrightarrow TGSW\\ (A, B) &\longmapsto A \boxtimes B = \begin{bmatrix} A \boxdot \mathbf{b_1} \\ \vdots \\ A \boxdot \mathbf{b_{(k+1)\ell}} \end{bmatrix}\\ (\mu_A, \mu_B) &\longmapsto \mu_A \cdot \mu_B\end{aligned}$$

Product



$$\|\mathsf{Err}(A \boxdot \mathbf{b})\|_\infty \leq \boxed{\ell' N \beta \eta_A + \|\mu_A\|_1 (1 + kN)\epsilon} + \boxed{\|\mu_A\|_1 \eta_{\mathbf{b}}}$$

where β and ϵ are the parameters used in the decomposition $Dec_{\mathbf{h},\beta,\epsilon}(\mathbf{b})$.

Table of contents

Faster bootstrapping

We applied our result to the fast bootstrapping proposed by Ducas and Micciancio (Eurocrypt 2015)

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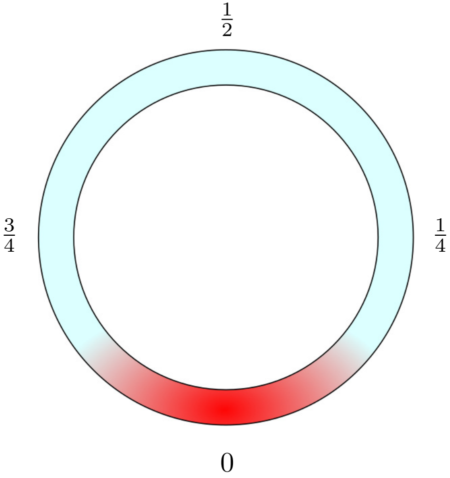
[DM15]: homomorphic NAND gate with fast bootstrapping in ~ 0.69 seconds

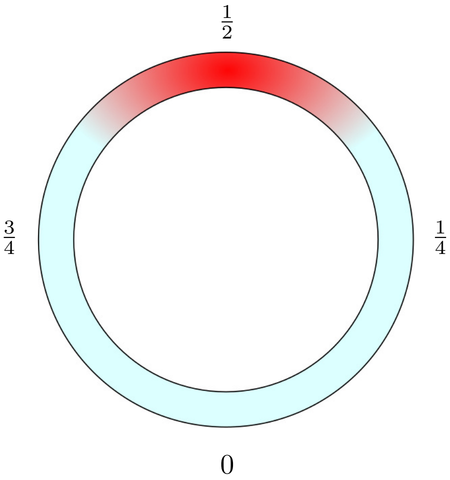
We replaced all the internal products in the bootstrapping procedure with the external one.

Result: (with further optimizations) we had a speed-up of a factor ~ 12 (bootstrapping in ~ 0.052 seconds)

Bootstrapping

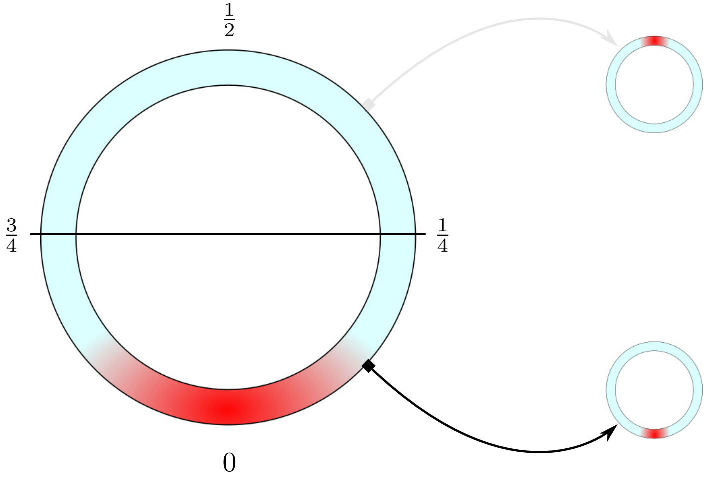


Bootstrapping



Bootstrapping

[Gentry09]-style bootstrap



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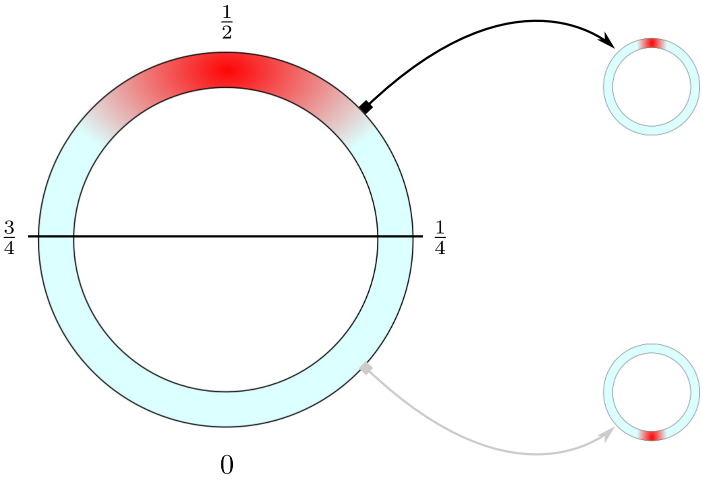

Bootstrapping

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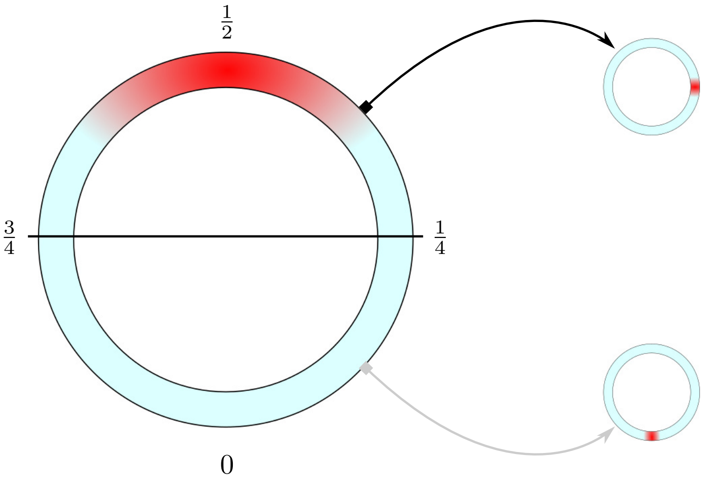
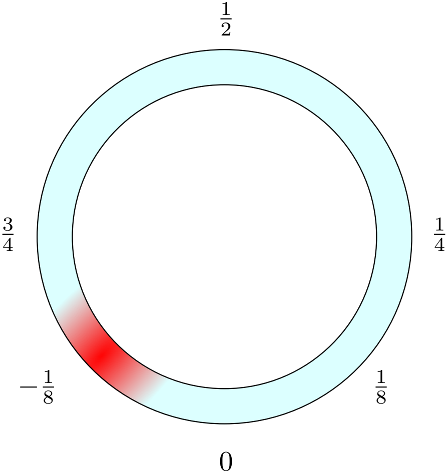


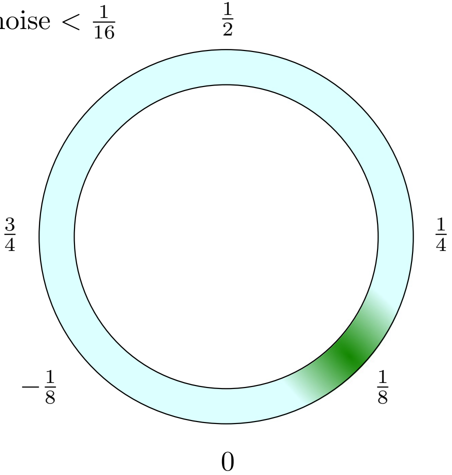
Gate Bootstrapping

false $:= \mathrm{LWE}(-\frac{1}{8})$, noise$< \frac{1}{16}$



Gate Bootstrapping

$\textcolor{green}{\text{true}} := \text{LWE}(+\frac{1}{8})$, noise $< \frac{1}{16}$



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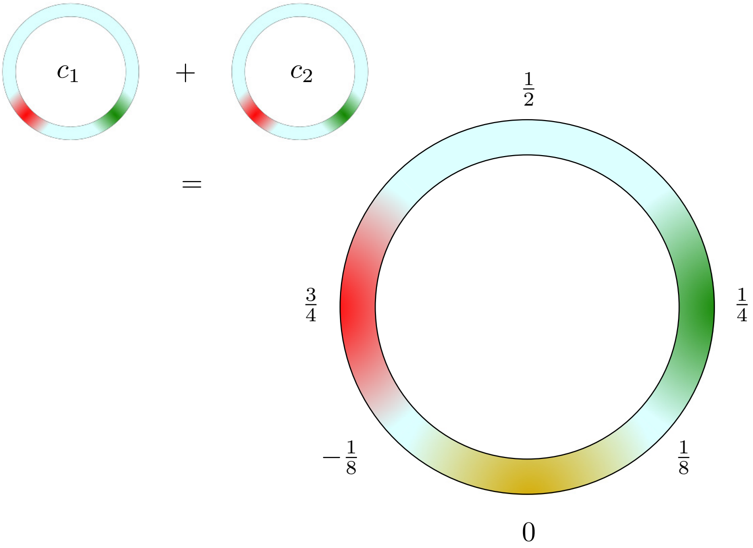

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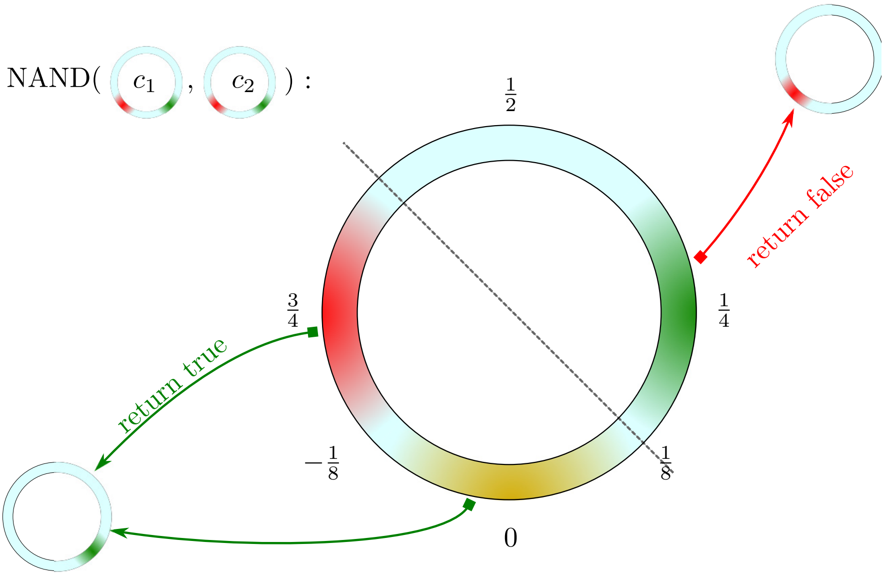

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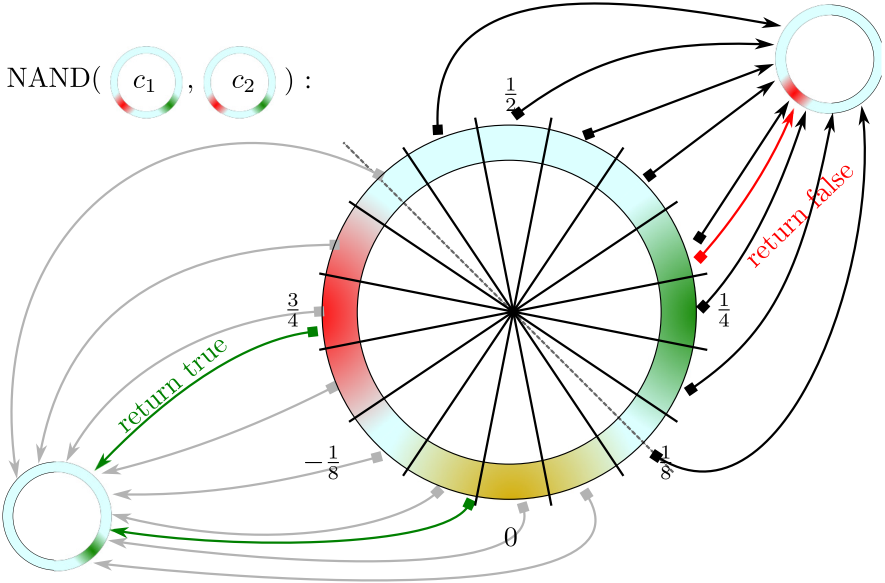
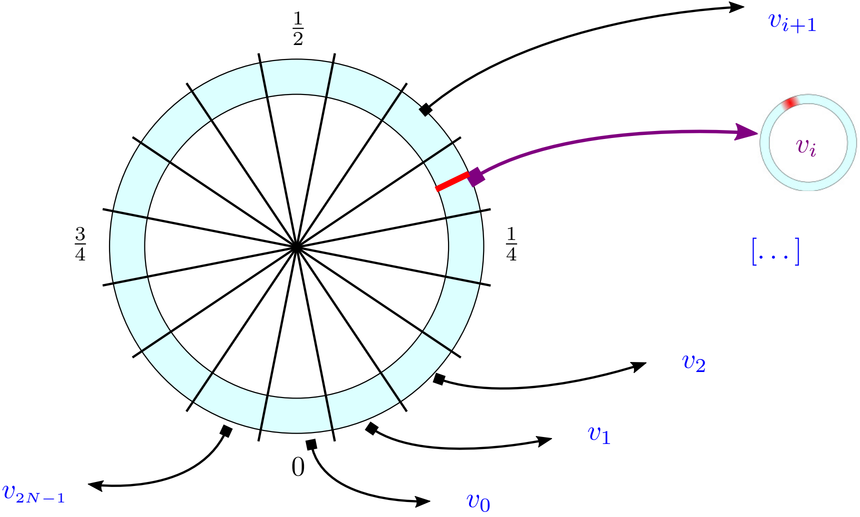


Gate Bootstrapping

[DM15/BR15]-(revisited)



Bootstrapping Algorithm (animation)

Bootstrapping algorithm of $(\mathbf{a}, b)$

1. Start from (a trivial) $\mathsf{TLWE}(v_0 + v_1 X + \cdots + v_{N-1} X^{N-1})$[a]
2. Rotate it by $p = -\varphi_{\mathbf{s}}(\mathbf{a}, b)$ positions
3. Extract the constant term (which encrypts v_p)

[a] N coefs mod $X^N + 1$ can be viewed as $2N$ coefs mod $X^{2N} - 1$ s.t. $v_{N+i} = -v_i$

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1. Multiply by X^{-b}

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2. For $i \in [1, n]$ multiply by $\mathsf{TGSW}(X^{-a_i s_i})$

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   - $X^{a_i s_i} = 1 + (X^{a_i} - 1) \cdot s_i$, with $s_i \in \{0, 1\}$

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   - $\mathsf{TGSW}(X^{a_i s_i}) = h + (X^{a_i} - 1) \cdot \mathsf{TGSW}(s_i)$, where $\mathrm{BK} = \mathsf{TGSW}(s_i)$

Security analysis

Security analysis

Numerical security estimates

Based on [APS15],[LP11],[DM15] results

1. Convert the instance to a lattice problem
   - ✔ we tested: UniqueSVP, red to SIS, modSwitch...
2. Apply the best heuristics
3. Optimized all non-relevant parameters: $m, \varepsilon, q,$ trials ...

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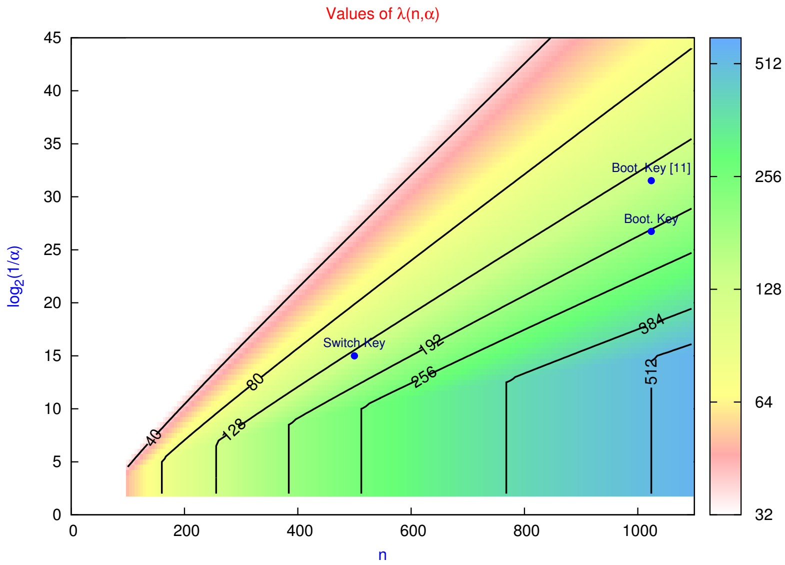
Important security parameters

1. Noise rate: α
2. Entropy of the secret: n

and that's all!

- λ expressed solely as a function of (n, α)

Security parameter - the rainbow



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TFHE implementation

https://tfhe.github.io/tfhe/

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- Before: 1 bootstrapping in 52 ms

TFHE implementation

https://tfhe.github.io/tfhe/

- Before: 1 bootstrapping in 52 ms
- Now: 1 bootstrapping in 20 ms

Conclusion

Summary

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Thank you!

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