Symmetric Encryption Scheme adapted to Fully Homomorphic Encryption Scheme: New Criteria for Boolean functions

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Outsourcing Computation

Alice

Limited storage
Limited power

Store ?
Compute ?

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Limited storage
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Store ✓
Compute ✓

Claude

Huge storage
Huge power

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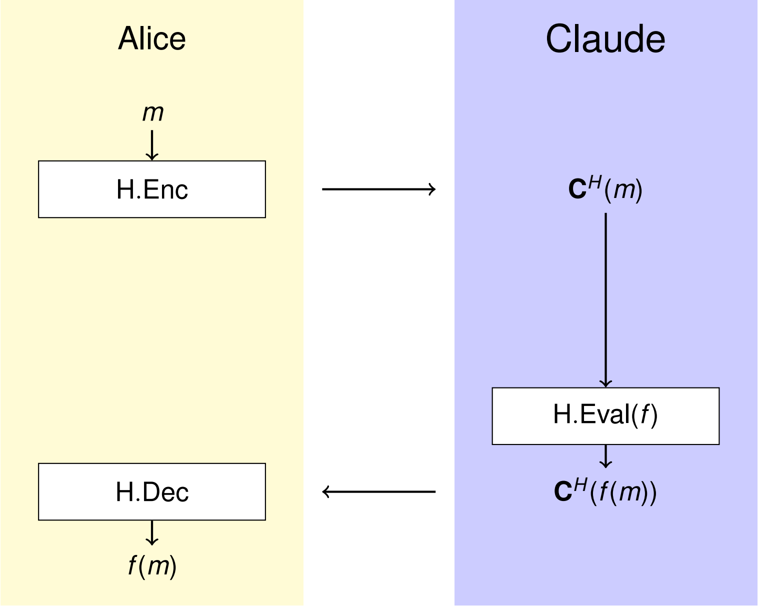
Bottlenecks:

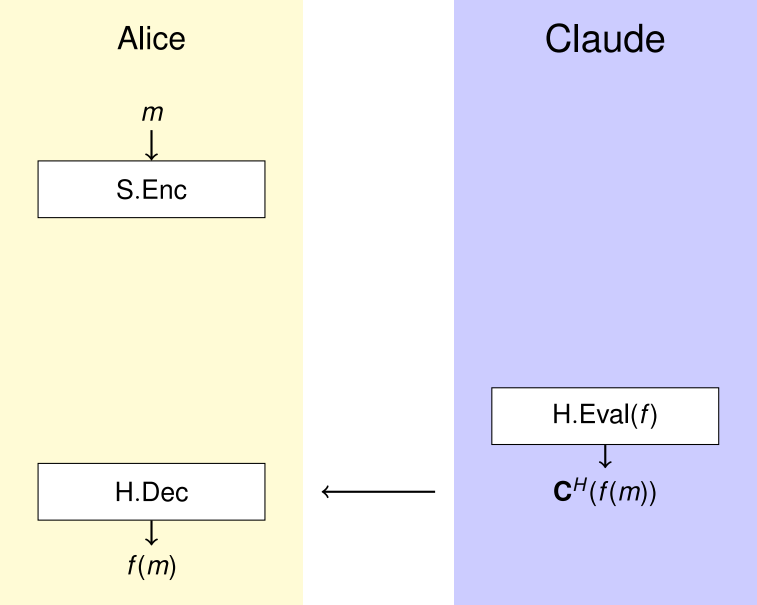
$\rightarrow$ high cost when high level of error

$\rightarrow$ high expansion factor

FHE Framework



SE-HE Hybrid Framework



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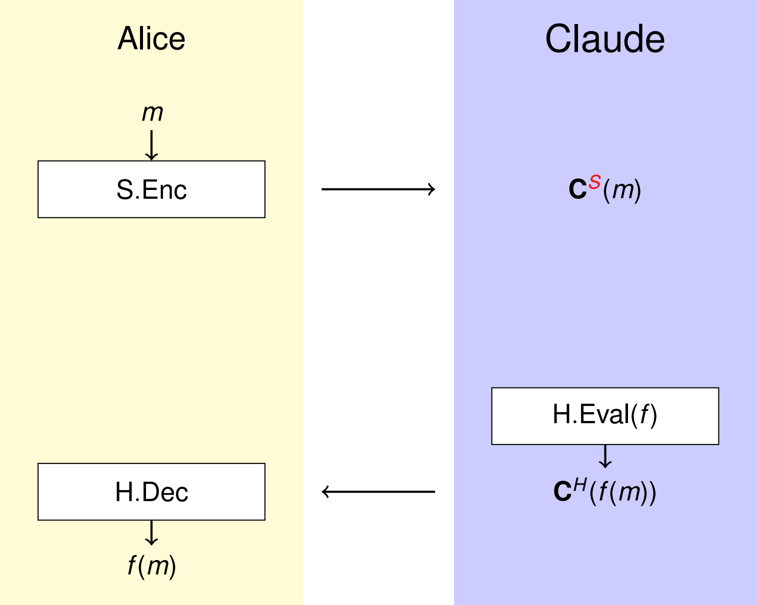

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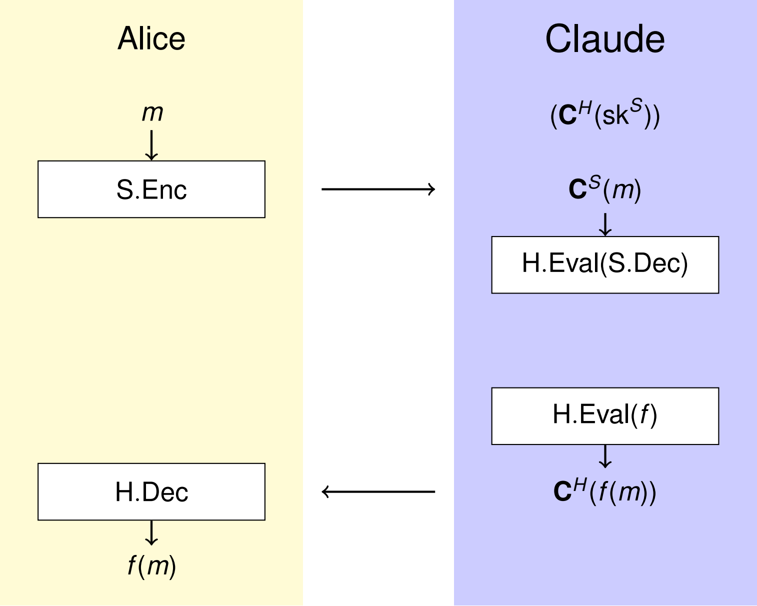

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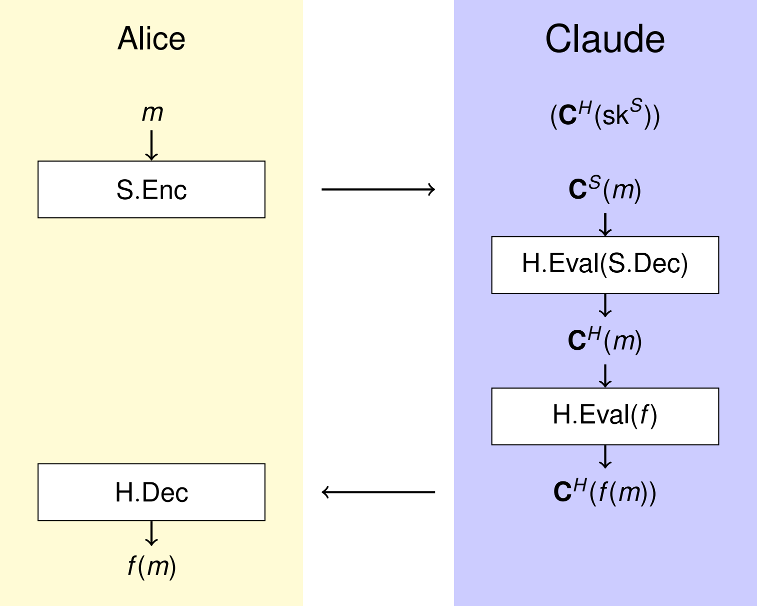

SE adapted to FHE

$\boxed{\text{H.Eval(S.Dec)}}$ as efficient as possible

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f in clear

$x_1 * x_2$

f in homomorphic

$\boxed{x_1} * \boxed{x_2}$

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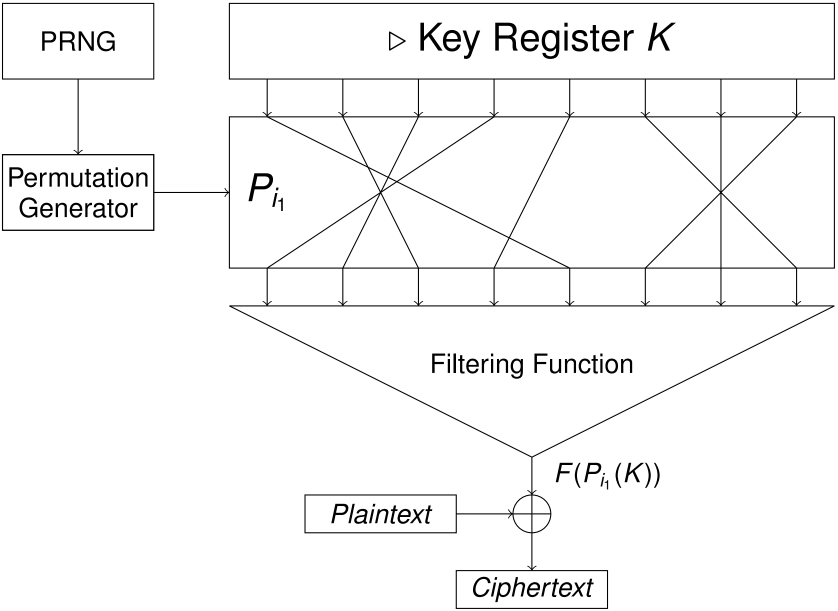
Optimize S.Dec circuit: Minimize homomorphic error growth

block cipher $\rightarrow$ too many rounds

stream cipher $\rightarrow$ increasing complexity

Summary

Filter Permutator: Construction



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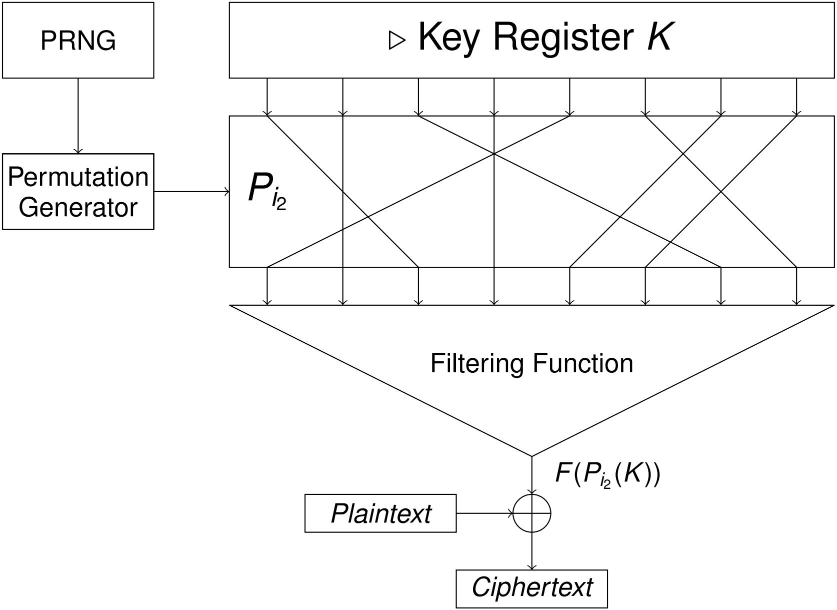

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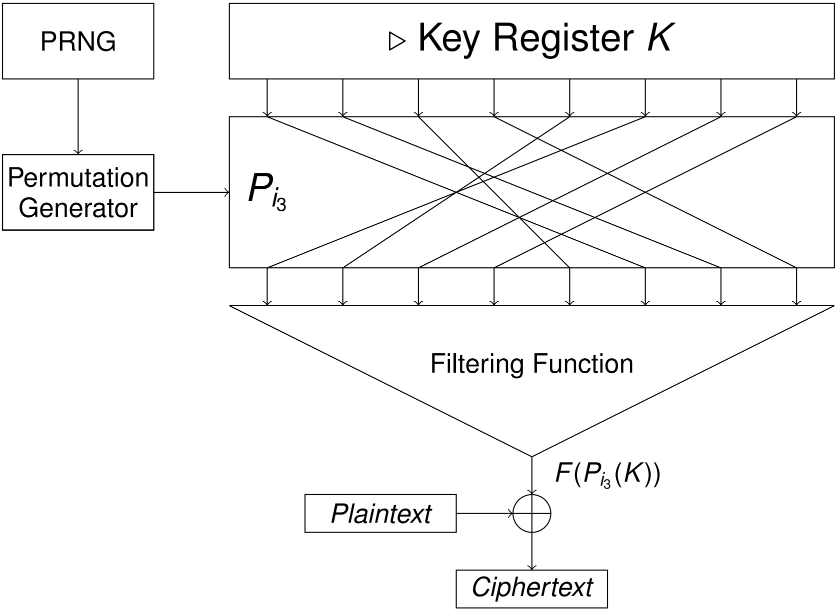
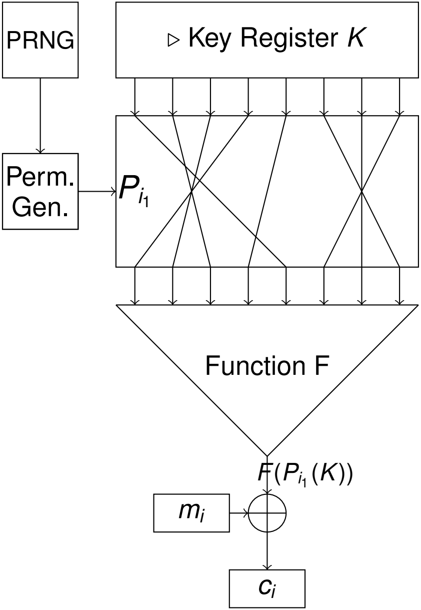


Filter Permutator: Homomorphic Evaluation



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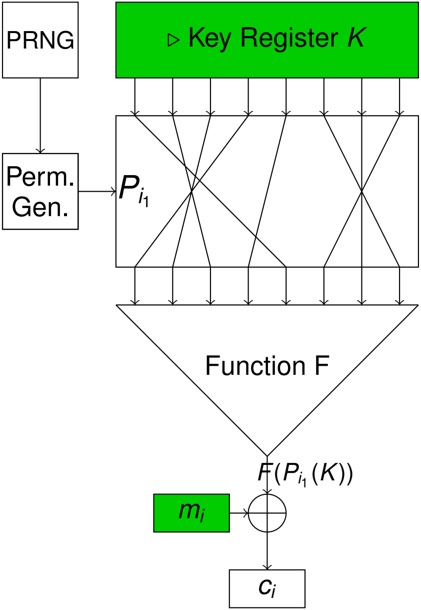

K_i, m_i: fresh

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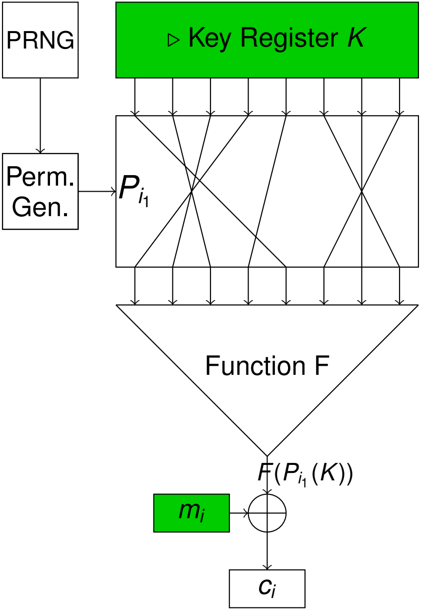

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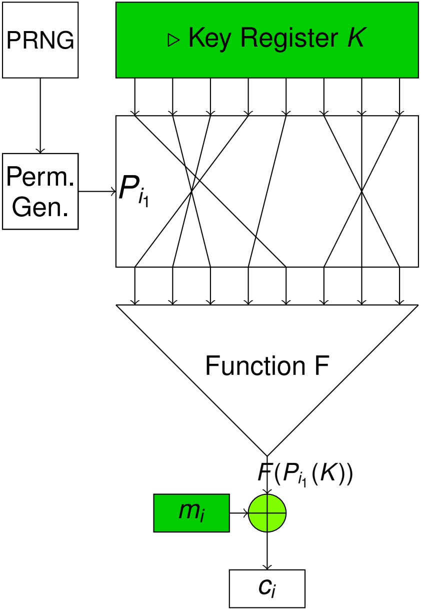

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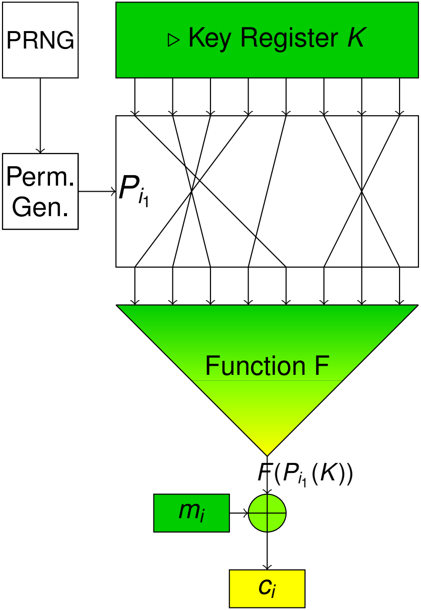

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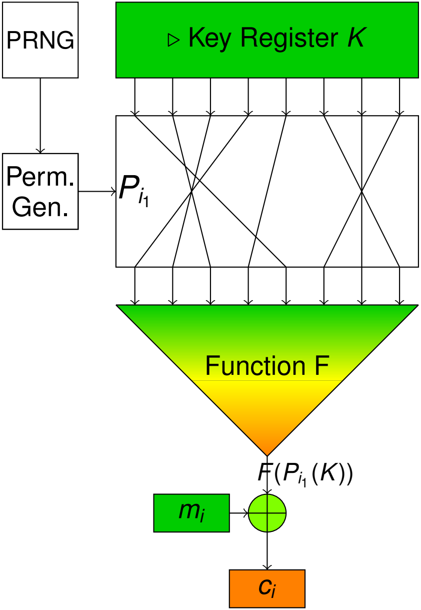

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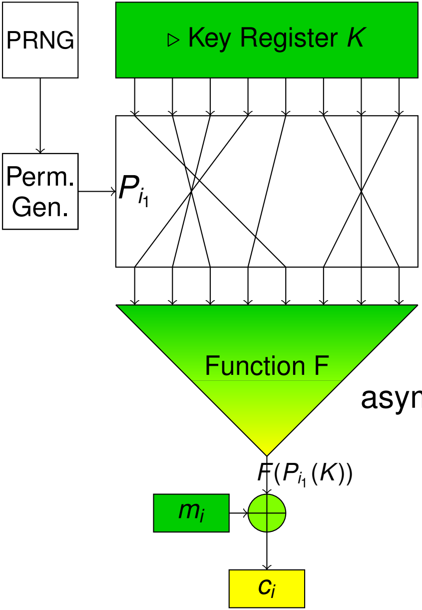

3*rd* generation FHE:

asymetric error growth for products

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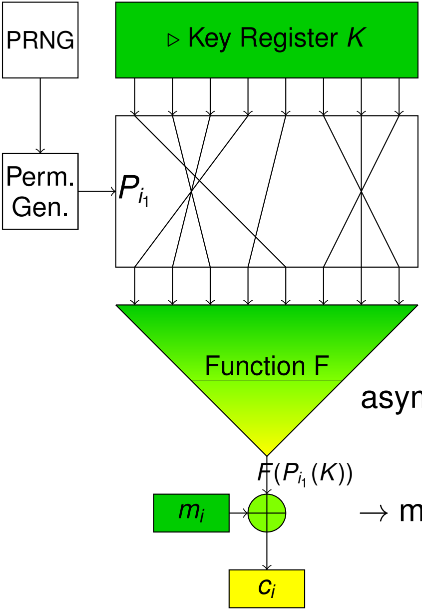

3*rd* generation FHE:

asymetric error growth for products

$\rightarrow$ additions

$\rightarrow$ multiplicative chains low noise ct

$\rightarrow$ few monomials

Summary

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Cryptanalysis Angle

"good" PRNG + "good" Shuffle $\approx$ random Permutations; what about F?

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Attacks on Filtering Function

- Algebraic
- Fast Algebraic
- Correlation
- High Order Correlation
- etc

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Standard Criteria

- Algebraic Immunity
- Fast Algebraic Immunity
- Resiliency
- Non Linearity

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Low cost constraints

- additions
- long multiplicative chains of simple functions
- few monomials

(Fast) Algebraic Attack

Algebraic Attack [CM03]

Let F be the keystream function of a stream cipher

1. find g a low algebraic degree function s.t. gF has low degree,
2. create T equations with monomials of degree $\leq deg(g)$,
3. linearize the system of T equations in $D = \sum_{i=0}^{deg(g)} \binom{N}{i}$ variables,
4. solve the system in $\mathcal{O}(D^{\omega})$.

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Algebraic Immunity

Let $F : \mathbb{F}_2^N \to \mathbb{F}_2$
we define

$$\begin{aligned} \mathrm{AI}(F) &= \min\{\max(\deg(g), \deg(gF), g \neq 0)\} \\ &= \{deg(g), g \neq 0 \mid gF = 0 \text{ or } g(F \oplus 1) = 0\} \end{aligned}$$

Attack complexity depends on $deg(g) \geq \mathrm{AI}(F)$

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Let F be the keystream function of a stream cipher

- find g and h low algebraic degree functions s.t. $gF = h$ with $\deg(g) < \mathrm{AI}(F)$ and possibly $deg(h) > deg(g)$,
- use codes methods to cancel monomials of degree higher than $deg(g)$,
- solve the system with better complexity than Algebraic Attack.

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we define $\mathrm{FAI}(F) = \min\{2\mathrm{AI}(F), \min_{1 \leq deg(g) \leq \mathrm{AI}(F)}\{deg(g) + deg(Fg), 3deg(g)\}\}$

Good Algebraic Immunity

(F)AI properties

upper bound:

$$\mathrm{AI}(F) \leq \lceil N/2 \rceil$$

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Majority function

$$x = (x_1, \cdots, x_N) \in \mathbb{F}_2^N, \quad Maj_N(x) = \begin{cases} 0 & \text{if } Hw(x) \leq \lfloor \frac{N}{2} \rfloor \\ 1 & \text{otherwise} \end{cases}$$

$N = 3$; $Maj_3(x) = x_1x_2 + x_1x_3 + x_2x_3$

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$\mathrm{AI}(Maj_N) = \lceil N/2 \rceil$
ANF: $\geq \binom{N}{\lceil N/2 \rceil}$ monomials

Low Cost and Good Algebraic Immunity

(F)AI properties

upper bound:

$$\mathsf{AI}(F) \leq \lceil N/2 \rceil$$

Direct sum property, $F(x_1, \cdots, x_N) = f_1(x_1, \cdots, x_\ell) + f_2(x_{\ell+1}, \cdots, x_N)$

$$\max(\mathsf{AI}(f_1), \mathsf{AI}(f_2)) \leq \mathsf{AI}(F) \leq \mathsf{AI}(f_1) + \mathsf{AI}(f_2)$$

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Triangular function

Let T_k be a Boolean function of $N = \frac{k(k+1)}{2}$ variables, built as the direct sum of k monomials of degree from 1 to k.

$T_4 = x_0 + x_1x_2 + x_3x_4x_5 + x_6x_7x_8x_9$

Low Cost and Good Algebraic Immunity

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$\mathrm{AI}(T_k) = k$
ANF: k monomials

Correlation Attack

Correlation attack/ BKW-like attack

Let F be the keystream function of a stream cipher

1. find g the best linear approximation of F,
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Nonlinearity

Let $F : \mathbb{F}_2^N \to \mathbb{F}_2$ and
we define $\mathrm{NL}(F) = \min_{g \text{ affine}} \{d_H(f, g)\}$,
where $d_H(f, g) = \#\{x \in \mathbb{F}_2^N \mid F(x) \neq g(x)\}$, the Hamming distance

The approximation error is $\frac{\mathrm{NL}(F)}{2^N}$.

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Balancedness

$F : \mathbb{F}_2^N \to \mathbb{F}_2$ is balanced if its output are uniformly distributed over $\{0, 1\}$

Resiliency

$F : \mathbb{F}_2^N \to \mathbb{F}_2$ is m resilient if any of its restrictions obtained by fixing at most m of its coordinates is balanced

Low Cost and good criteria

direct sum properties

Let F be the direct sum of f_1 in n_1 variables and f_2 in n_2 variables

- $\mathrm{res}(f) = \mathrm{res}(f_1) + \mathrm{res}(f_2) + 1$,
- $\mathrm{NL}(F) = 2^{n_2}\mathrm{NL}(f_1) + 2^{n_1}\mathrm{NL}(f_2) - 2\mathrm{NL}(f_1)\mathrm{NL}(f_2)$

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Low cost functions

- Resiliency:
  $L_n = \sum_{i=1}^{n} x_i$; $n-1$ resilient
- Nonlinearity:
  $Q_{\frac{n}{2}} = \sum_{i=1}^{\frac{n}{2}} x_{2i-1}x_{2i}$
- Algebraic Immunity:
  $T_k = \sum_{i=1}^{k} \prod_{j=1}^{i} x_{\frac{i(i+1)}{2}+j}$

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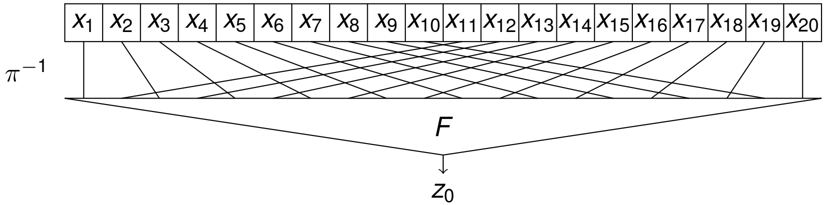
- Algebraic Immunity:
  $T_k = \sum_{i=1}^{k} \prod_{j=1}^{i} x_{\frac{i(i+1)}{2}+j}$
- Low cost and optimized criteria:
  $F = L_{n_1} + Q_{\frac{n_2}{2}} + T_k$

Summary

Guess and Determine Attacks



$$
\begin{aligned}
z_0 = {} & x_{\pi(1)} + x_{\pi(2)} + x_{\pi(3)} + x_{\pi(4)} \\
+ {} & x_{\pi(5)}x_{\pi(6)} + x_{\pi(7)}x_{\pi(8)} + x_{\pi(9)}x_{\pi(10)} \\
+ {} & x_{\pi(11)} + x_{\pi(12)}x_{\pi(13)} + x_{\pi(14)}x_{\pi(15)}x_{\pi(16)} + x_{\pi(17)}x_{\pi(18)}x_{\pi(19)}x_{\pi(20)}
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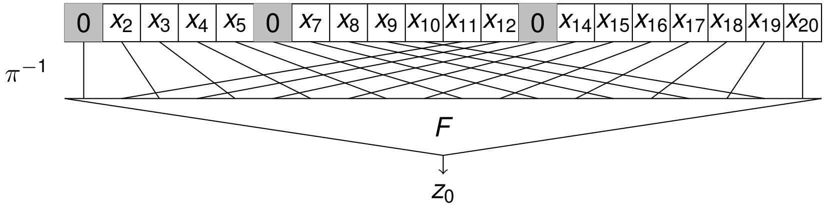
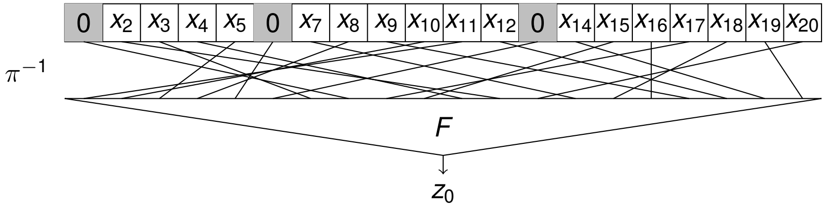


$$\begin{aligned} z_0 &= \cancel{x_{\pi(1)}} + x_{\pi(2)} + x_{\pi(3)} + x_{\pi(4)} \\ &+ \cancel{x_{\pi(5)}x_{\pi(6)}} + x_{\pi(7)}x_{\pi(8)} + x_{\pi(9)}x_{\pi(10)} \\ &+ \cancel{x_{\pi(11)}} + x_{\pi(12)}x_{\pi(13)} + x_{\pi(14)}x_{\pi(15)}x_{\pi(16)} + x_{\pi(17)}x_{\pi(18)}x_{\pi(19)}x_{\pi(20)} \end{aligned}$$

Guess & Determine attack [DLR16]

- Guess ℓ positions being 0,

Guess and Determine Attacks



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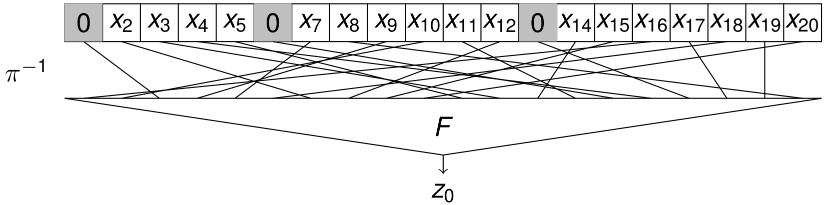

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Guess & Determine attack [DLR16]

- Guess ℓ positions being 0,
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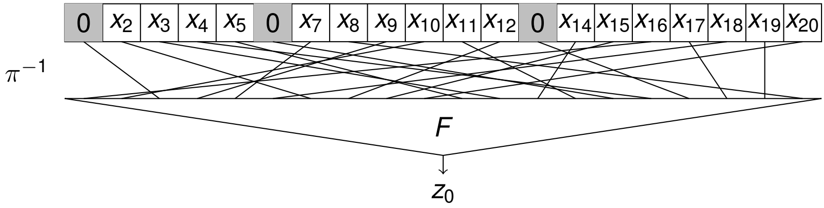
- collect all degree 2 equations,

Guess and Determine Attacks



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z_0 \;=\;& x_{\pi(1)} + x_{\pi(2)} + \cancel{x_{\pi(3)}} + x_{\pi(4)} \\
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+\;& x_{\pi(11)} + x_{\pi(12)}x_{\pi(13)} + \cancel{x_{\pi(14)}x_{\pi(15)}x_{\pi(16)}} + \cancel{x_{\pi(17)}x_{\pi(18)}x_{\pi(19)}x_{\pi(20)}}
\end{aligned}
$$

Guess & Determine attack [DLR16]

- Guess ℓ positions being 0,
- focus on permutations cancelling the monomials of degree > 2,
- collect all degree 2 equations,
- linearise and try to solve the system,
- time complexity $2^{\ell}(1 + N + \binom{N}{2})^{\omega}$, data complexity $1/Pr(P)$.

G&D attacks and new Boolean criteria

Attack lessons

- zero cost homomorphic update $\rightarrow$ unchanged key bits,
- ℓ guesses $\rightarrow$ F restricted to F' on $N - \ell$ variables,
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Recurrent criteria

Recurrent AI; $\mathrm{AI}[\ell](F)$:
$\mathrm{AI}[\ell](F)$ is the minimal algebraic immunity over all functions obtained by fixing ℓ variables of F.

Similarly,
$\mathrm{FAI}[\ell](F), \mathrm{NL}[\ell](F)$,and $\mathrm{res}[\ell](F)$

Recurrent Algebraic immunity

Recurrent AI; AI[ℓ](F)

AI[ℓ](F) is the minimal algebraic immunity over all functions obtained by fixing ℓ variables of F.
example:
$\mathrm{AI}[1](f(x_1, x_2)) = \min[\mathrm{AI}(f(0, x_2)), \mathrm{AI}(f(1, x_2)), \mathrm{AI}(f(x_1, 0)), \mathrm{AI}(f(x_1, 1))]$

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For all Boolean function F:

$$\mathrm{AI}(F) - \ell \leq \mathrm{AI}[\ell](F) \leq \mathrm{AI}(F)$$

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upper bound: g defining AI(F); a guess where g is not null.
lower bound: hypothesis $\mathrm{AI}[1](F) < \mathrm{AI}(F) - 1$ leads to contradiction

Recurrent Algebraic immunity

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Majority function, $\ell = 2$

$$x = (x_1, \cdots, x_N) \in \mathbb{F}_2^N, \quad Maj_N(x) = \begin{cases} 0 & \text{if } Hw(x) \leq \lfloor \frac{N}{2} \rfloor \\ 1 & \text{otherwise} \end{cases}$$

$F = Maj_N$;
$\mathrm{AI}(F) = \lceil N/2 \rceil$

$$\lceil N/2 \rceil - 2 \leq \mathrm{AI}[2](F) \leq \lceil N/2 \rceil$$

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$$\mathrm{AI}(F) - \ell \leq \mathrm{AI}[\ell](F) \leq \mathrm{AI}(F)$$

Majority function, $\ell = 2$, fixing $x_1 = 1$,and $x_2 = 0$

$$\bar{x} = (x_3, \cdots, x_N) \in \mathbb{F}_2^{N-2}, \quad F'(\bar{x}) = \begin{cases} 0 & \text{if } Hw(x) \leq \lfloor \frac{N}{2} \rfloor - 1 \\ 1 & \text{otherwise} \end{cases}$$

$F' = Maj_{N-2}$;
$\mathrm{AI}(F') = \lceil (N-2)/2 \rceil$

$$\lceil N/2 \rceil - 2 \leq \mathrm{AI}[2](F) \leq \lceil N/2 \rceil - 1$$

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$(F' + 1) \cdot S_{\lceil (N-4)/2 \rceil} = 0;$
$\mathrm{AI}(F') = \lceil (N-4)/2 \rceil$

$$\lceil N/2 \rceil - 2 = \mathrm{AI}[2](F)$$

Recurrent Criteria and Direct Sums of Monomials

Criteria for Direct Sums of Monomials

F direct sum of monomials $\leftrightarrow$ vector $\mathbf{m}_F = [m_1, m_2, \cdots, m_k]$
Example: T_4; $\mathbf{m}_{T_4} = [1, 1, 1, 1]$

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Two recurrent criteria:

- $\mathbf{m}_F^*$ the number of nonzero values of $\mathbf{m}_F$,
- $\delta_{\mathbf{m}_F} = \frac{1}{2} - \frac{\mathrm{NL}(F)}{2^N}$; "bias".

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Criteria bounds

For all choice of ℓ fixed variables, $F[\ell]$ follows these properties

- $\sum_{i=1}^{\deg(F[\ell])} m_i[\ell] \geq (\sum_{i=1}^{\deg(F)} m_i) - \ell$,
- $\mathbf{m}_{F[\ell]}^* \geq \mathbf{m}_F^* - \lfloor \frac{\ell}{\min_{1 \leq i \leq \deg(F)} m_i} \rfloor$,
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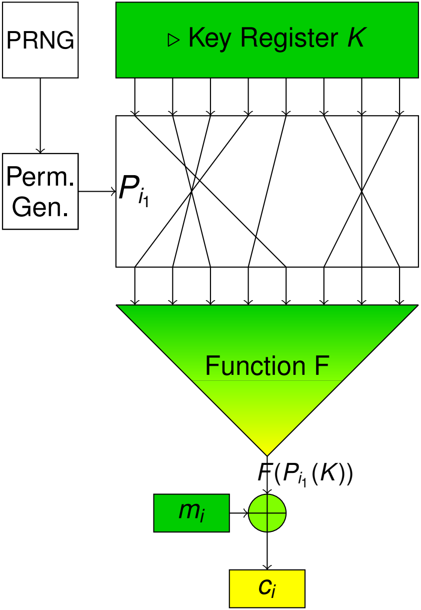
Concrete bounds for (fast) algebraic attacks and correlation attacks for all ℓ:

$$\mathbf{m}_{F[\ell]}^* \leftrightarrow \text{upper bound on } \mathrm{AI}[\ell](F),$$
$$\delta_{\mathbf{m}_{F[\ell]}} \leftrightarrow \text{upper bound on } \mathrm{NL}[\ell](F).$$

Summary

Filter Permutator: Hamming weight of F input



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$$Im(\psi) \subsetneq \mathbb{F}_2^N$$

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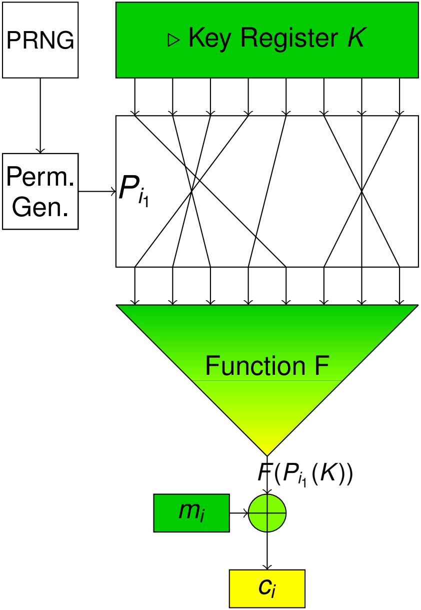

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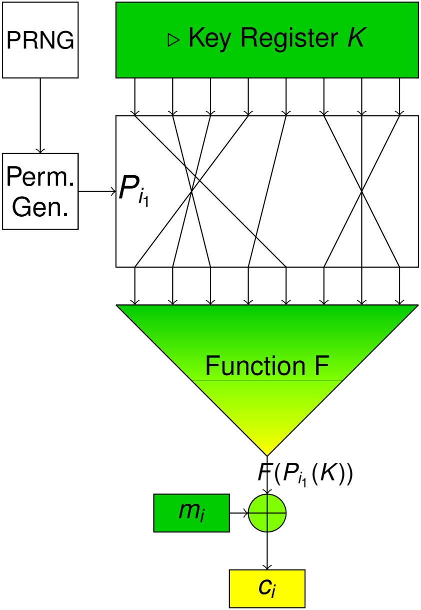

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$$E_{N,k} := \{x \mid \mathrm{w_H}(x) = k\}$$

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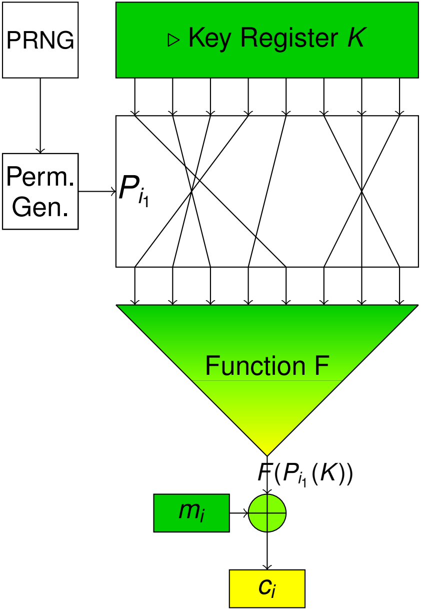

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$\rightarrow$ balancedness
$\rightarrow$ non-linearity
$\rightarrow$ algebraic immunity

Balancedness on constant Hamming weight input

$S_1 = x_1 + x_2 + \cdots + x_n \quad ; n-1$ resilient but constant for all k

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Weightwise Perfectly Balanced Function

Boolean function f defined over $\mathbb{F}_2^n$, is *weightwise perfectly balanced* (*WPB*):

$$\forall k \in [1, n-1], \mathrm{w_H}(f)_k = \frac{\binom{n}{k}}{2}, \text{ and, } f(0, \ldots, 0) = 0; \quad f(1, \ldots, 1) = 1.$$

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Secondary Construction of *WPB* Functions

From f, f', and g, 3 n-variable *WPB* functions and g' n-variable arbitrary function we build a $2n$-variable *WPB* function:

$$h(x, y) = f(x) + \prod_{i=1}^{n} x_i + g(y) + (f(x) + f'(x))g'(y)$$

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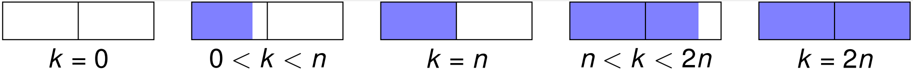

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$k = 0$ $\quad$ $0 < k < n$ $\quad$ $k = n$ $\quad$ $n < k < 2n$ $\quad$ $k = 2n$

case $k = 0$

$\mathrm{w_H}(x) = 0 \qquad \mathrm{w_H}(y) = 0$

$$f(0,\cdots,0) = g(0,\cdots,0) = f'(0,\cdots,0) = 0$$

h(0,0)=0

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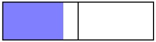

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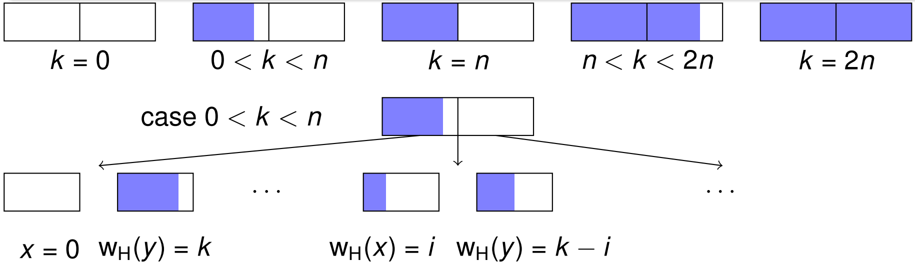

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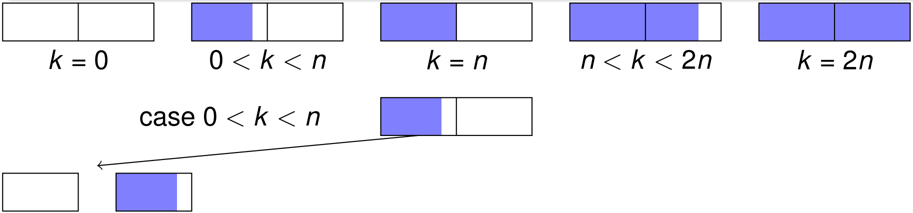

$x = 0$ $\mathrm{w_H}(y) = k$

case $x = 0$ $\qquad f(0, \cdots, 0) = f'(0, \cdots, 0) = 0$

h(0,y)=g(y) $\qquad$ g balanced on $E_{n,k}$

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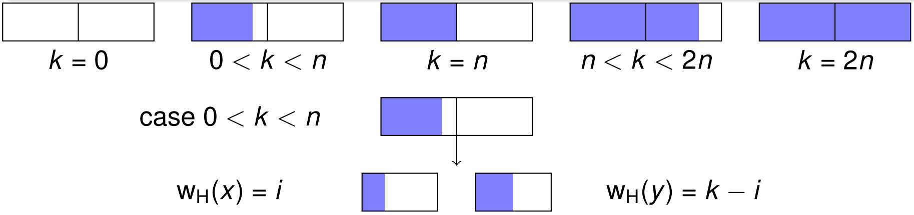

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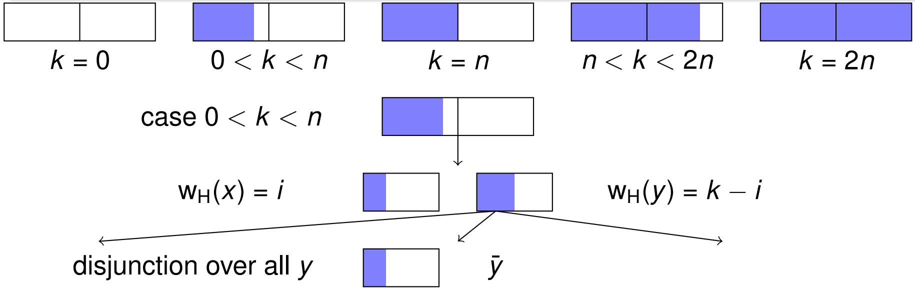

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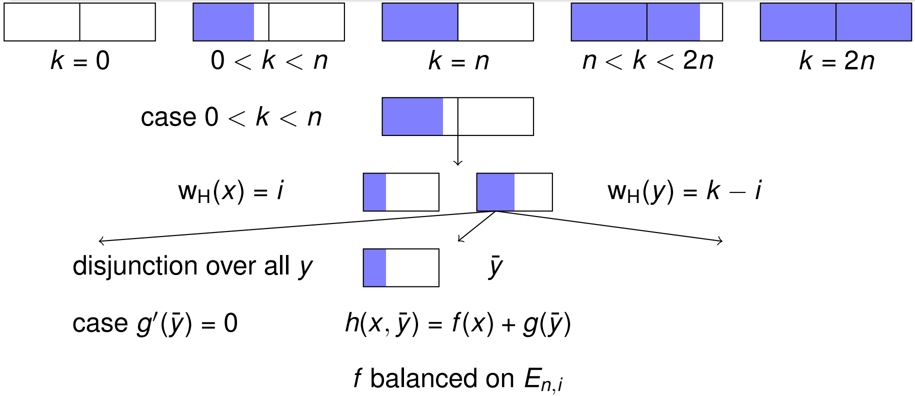

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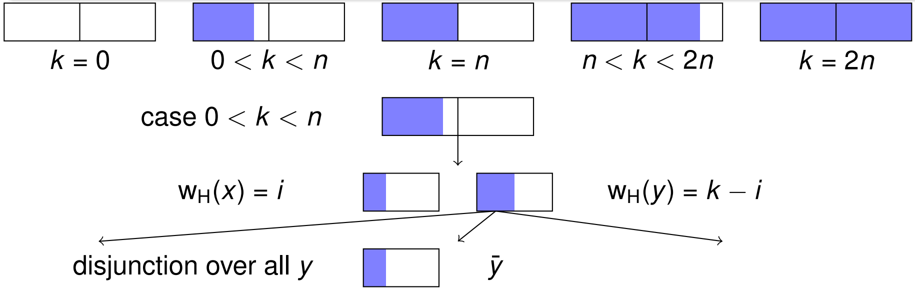

case $g'(\bar{y}) = 1$ $\quad$ $h(x, \bar{y}) = f'(x) + g(\bar{y})$

f' balanced on $E_{n,i}$

Restricted non-linearity

Non-linearity over E

Let $E \subset \mathbb{F}_2^n$ and f any Boolean function defined over E.
$\mathrm{NL}_E(f) = \min_g\{d_H(f, g) \text{ over } E\}$, where g is an affine function over $\mathbb{F}_2^n$.

Upper bound on NL_E

For every subset E of $\mathbb{F}_2^n$ and every Boolean function f defined over E, we have:

$$\mathrm{NL}_E(f) \leq \frac{|E|}{2} - \frac{\sqrt{|E|}}{2}.$$

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$$\begin{aligned}
\sum_{a \in \mathbb{F}_2^n} \left(\sum_{x \in E} (-1)^{f(x)+a\cdot x} \right)^2 &= \sum_{x,y \in E} (-1)^{f(x)+f(y)} \sum_{a \in \mathbb{F}_2^n} (-1)^{a\cdot(x+y)} \\
&= ?
\end{aligned}$$

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$$\text{if } x + y \neq 0, \quad \sum_{a \in \mathbb{F}_2^n} (-1)^{a \cdot (x+y)} = 0$$

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maximum always greater than mean; $\max \geq \sqrt{|E|}$.

Restricted non-linearity

Better upper bound on NL_E

For every subset E of $\mathbb{F}_2^n$ and every Boolean function f defined over E, we have:

$$\mathrm{NL}_E(f) \leq \frac{|E|}{2} - \frac{\sqrt{|E + \lambda|}}{2}.$$

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maximum always greater than mean; $\max \geq \sqrt{|E|}$.

Restricted non-linearity

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λ can be assumed > 0 for some cases, in particular $\mathrm{NL}_{E_{n,k}}(f) < \frac{\binom{n}{k}}{2} - \frac{\sqrt{\binom{n}{k}}}{2}$.

Non-linearity degradation

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For all even $n \geq 4$ there exists quadratic bent functions such that $\forall k$, $\mathrm{NL}_k = 0$.

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Bent functions and simplectic form [Car10]

f with associated simplectic form; $(x, y) \to f(x, y) + f(x) + f(y) + f(0)$ is bent iff the kernel $E = \{x \in F_2^n; \forall y \in F_2^n, f(x + y) + f(x) + f(y) + f(0) = 0\}$ is equal to $\{0\}$.

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Restricted algebraic immunity

Algebraic immunity over E

Let f defined over a set E:

$$\mathrm{AI}_E(f) = \min\{\deg(g); g \cdot f = 0 \text{ or } g \cdot (f+1) \text{ over } E \text{ and } g \neq 0 \text{ over } E\}.$$

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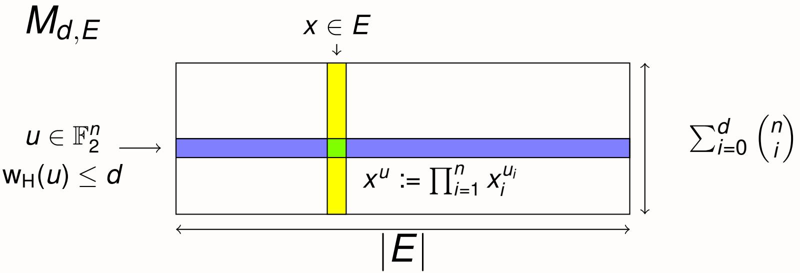
Upper bound on algebraic immunity



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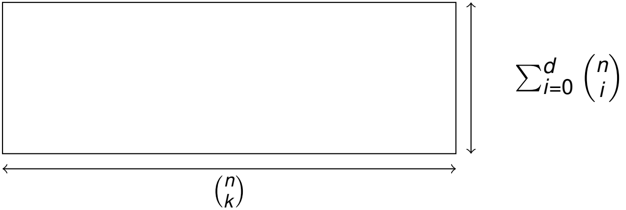

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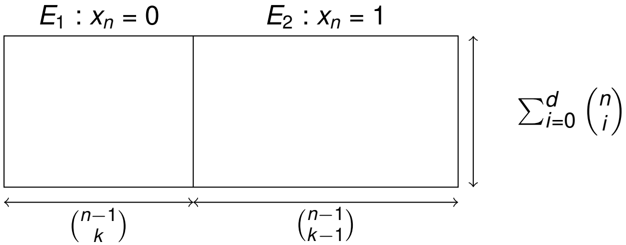

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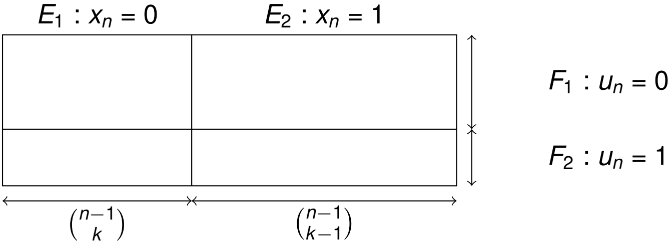

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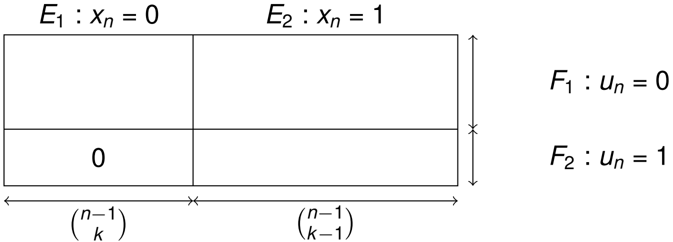

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\hline
F_1 : u_n = 0 & M_{n-1,d,k} & \\
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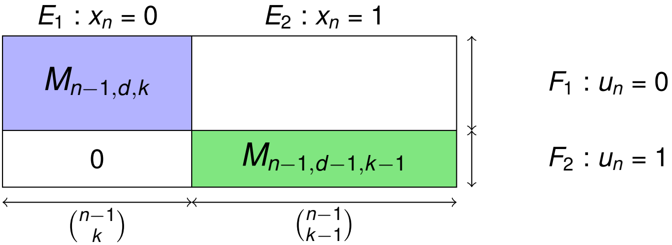

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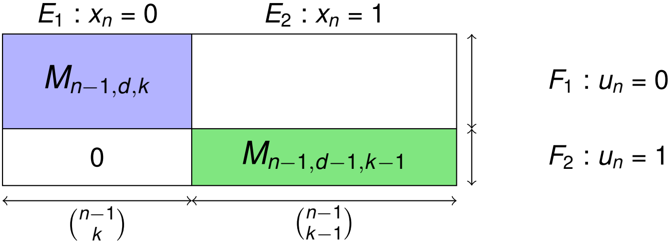

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initialisation: $d \geq k$ or $d \geq n-k$ gives canonical base; $\operatorname{rank}(M) = \binom{n}{k}$

Algebraic immunity degradation

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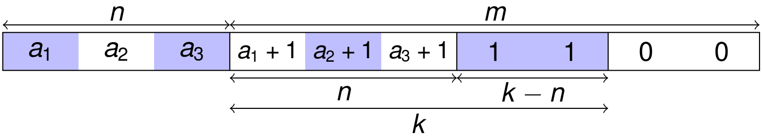

Algebraic immunity degradation

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n			m						
a_1	a_2	a_3	$a_1 + 1$	$a_2 + 1$	$a_3 + 1$	1	1	0	0
			n			$k - n$			
			k						

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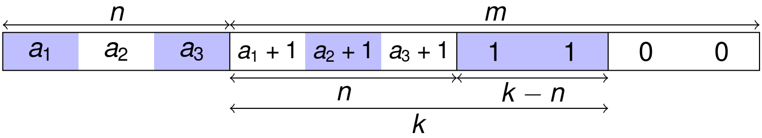

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Example of direct sum reaching the bound

$$f(x_1, x_2, x_3) = x_1 + x_2 x_3, \quad \mathrm{AI}(f) = 2$$

$$g(x_4, x_5, x_6) = x_4 + x_5 + x_6, \quad \deg(g) = 1$$

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$$x_2(f + g) = x_2(1 + \sum_{i=1}^{6} x_i) = x_2(1 + S_1)$$

$$S_1(x) = 1 \text{ for odd k } \Rightarrow \mathrm{AI}_3(F) = 1$$

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- higher number of variables with simpler circuit,
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Thanks for your attention!