Multiresolution Analysis of Incomplete Rankings

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Outline

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Motivations

Why group-based harmonic analysis on $\mathfrak{S}_n$ is not adapted

Our results

The mathematical construction

Ongoing research

Our purpose

Model ranking data.

General types of ranking data

Complete set of items $[\![n]\!] = \{1, \ldots, n\}$ (no features).

- Full rankings:

$$a_1 \prec \cdots \prec a_n$$

- Partial rankings:

$$a_{1,1}, \ldots, a_{1,n_1} \prec \cdots \prec a_{r,1}, \ldots, a_{r,n_r} \qquad \text{with } \sum_{i=1}^{r} n_i = n$$

- Incomplete rankings:

$$a_1 \prec \cdots \prec a_k \qquad \text{with } k < n$$

The case of full rankings

Full ranking $a_1 \prec \ldots \prec a_n \leftrightarrow$ **Permutation** σ **such that** $\sigma(a_i) = i$

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Statistical setting for observations: $\sigma_1, \ldots, \sigma_N \sim p$ i.i.d.
where p is a probability distribution on $\mathfrak{S}_n$,

$$p \in L(\mathfrak{S}_n) := \{f : \mathfrak{S}_n \to \mathbb{R}\}.$$

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Many approaches to characterize the structure of p

- Parametric modeling: Placket-Luce model, Mallows model, Thurstone model, ...
- "Non-parametric" modeling: Kernel-based smoothing, Independence assumptions, **Harmonic analysis**, ...

Group-based harmonic analysis on $\mathfrak{S}_n$

- For $\sigma \in \mathfrak{S}_n$, define the translation operator

$$\begin{aligned} T_\sigma : L(\mathfrak{S}_n) &\to L(\mathfrak{S}_n) \\ f &\mapsto f(\sigma^{-1}.) \end{aligned}$$

- The T_σ's do not commute but $\sigma \mapsto T_\sigma$ is the left regular representation of $\mathfrak{S}_n$
- Hence the spectral decomposition

$$L(\mathfrak{S}_n) \cong \bigoplus_{\lambda \vdash n} d_\lambda S^\lambda$$

where

  - the λ's correspond to frequencies
  - the S^λ are irreducible representations of $\mathfrak{S}_n$
  - $d_\lambda = \dim S^\lambda$ and is also its multiplicity in the decomposition

Group-based harmonic analysis on $\mathfrak{S}_n$

- $\lambda \vdash n$ is a *partition of n*: $\lambda = (\lambda_1, \ldots, \lambda_r) \in [\![n]\!]^r$ with $\lambda_1 \geq \cdots \geq \lambda_r$ and such that $\sum_{i=1}^{r} \lambda_i = n$.

Group-based harmonic analysis on $\mathfrak{S}_n$

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- Dominance ordering on partitions of n: for $\lambda = (\lambda_1, \dots, \lambda_r)$ and $\mu = (\mu_1, \dots, \mu_s)$ define

$$\lambda \trianglerighteq \mu \text{ if for all } j \in \{1, \dots, r\}, \sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i.$$

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$$\lambda \trianglerighteq \mu \ \text{ if for all } j \in \{1, \dots, r\},\ \sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i.$$

- The nested sequence of subspaces

$$S^{(n)} \subset S^{(n)} \oplus S^{(n-1,1)} \subset \cdots \subset \bigoplus_{\lambda \trianglerighteq \lambda_0} S^\lambda \subset \cdots \subset \bigoplus_{\lambda \trianglerighteq 1^n} S^\lambda = L(\mathfrak{S}_n)$$

defines a *meaningful* approximation procedure for $L(\mathfrak{S}_n)$.

General types of ranking data

Items	Ranks	Ranking form
Full rankings		
$[\![n]\!]$	$[\![n]\!]$	$a_1 \prec \cdots \prec a_n$ $\sigma : [\![n]\!] \rightarrow [\![n]\!]$ (bijective) $\sigma^{-1}(i) = a_i$
Partial rankings		
$[\![n]\!]$	$\{1, \ldots, r\}$	$a_{1,1}, \ldots, a_{1,n_1} \prec \cdots \prec a_{r,1}, \ldots, a_{r,n_r}$ $\gamma : [\![n]\!] \rightarrow \{1, \ldots, r\}$ (surjective) $\gamma^{-1}(\{i\}) = \{a_{i,1}, \ldots, a_{i,n_i}\}$
Incomplete rankings		
$A \subset [\![n]\!]$ $\lvert A\rvert \geq 2$	$\{1, \ldots, \lvert A\rvert\}$	$a_1 \prec \cdots \prec a_{\lvert A\rvert}$ $\pi : A \rightarrow \{1, \ldots, \lvert A\rvert\}$ (bijective) $\pi^{-1}(i) = a_i$

Application

Example: Recommendation system

- $[\![n]\!]$ represents the catalog of items (movie, songs, books, ...)
- Each user expresses preferences on subsets of items of the form
$$a_1 \prec \cdots \prec a_k,$$
with $k \leq k_0$
- **Knowing the preferences of a given user, in what order should we present a subset of items?**

Principle

n **is big**, of the order of 10^6
k_0 **is small**, of the order of 10 ($k_0 = 2$ for pairwise comparisons)

Formalism

Setting

- $[\![n]\!]$: complete set of items
- $\mathcal{A} \subset \mathcal{P}([\![n]\!])$: observation design
- $(P_A)_{A\in\mathcal{A}}$: family of probability distributions on each $\mathfrak{R}_A$

Notations

For a finite set E,

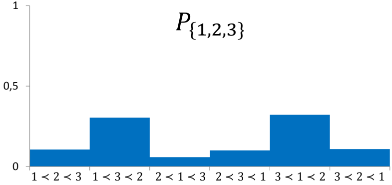
$$\mathcal{P}(E) := \{A \subset E \mid |A| \geq 2\}$$

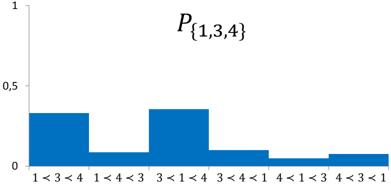
For $A \in \mathcal{P}([\![n]\!])$,

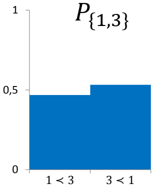
$$\mathfrak{R}_A := \{\pi : A \to \{1, \ldots, |A|\} \mid \pi \text{ bijective}\}\text{: rankings on } A$$

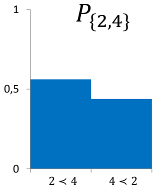
Example

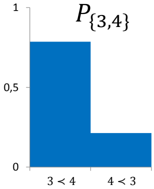
$n = 4$ and $\mathcal{A} = \{\{1,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}\}$











Consistency assumption

Data arise in the space

$$(P_A)_{A\in\mathcal{A}} \in \bigoplus_{A\in\mathcal{A}} L(\mathfrak{R}_A).$$

Consistency assumption: $(P_A)_{A\in\mathcal{A}}$ is a sub-family of a family $(P_A)_{A\in\mathcal{P}(\llbracket n\rrbracket)}$ that satisfies for any $A = \{a_1, \ldots, a_k\} \in \mathcal{P}(\llbracket n\rrbracket)$ with $k < n$ and $b \in \llbracket n\rrbracket \setminus A$,

$$P_A(a_{i_1} \prec \ldots \prec a_{i_k}) = P_{A\cup\{b\}}(a_{i_1} \prec \ldots \prec a_{i_k} \prec b) + \ldots$$
$$+ P_{A\cup\{b\}}(a_{i_1} \prec b \prec \ldots \prec a_{i_k}) + P_{A\cup\{b\}}(b \prec a_{i_1} \prec \ldots \prec a_{i_k}). \quad (*)$$

Example

$P_{\{1,3\}}(1 \prec 3) = P_{\{1,2,3\}}(1 \prec 3 \prec 2) + P_{\{1,2,3\}}(1 \prec 2 \prec 3) + P_{\{1,2,3\}}(2 \prec 1 \prec 3)$

Consistency assumption

Assumption $(*)$ $\Leftrightarrow$ There exists p probability distribution on $\mathfrak{S}_n$ such that for all $A \in \mathcal{A}$,

$$P_A(\pi) = \sum_{\sigma \in \mathfrak{S}_n(\pi)} p(\sigma) \quad \text{for all } \pi \in \mathfrak{R}_A,$$

where

$$\mathfrak{S}_n(\pi) = \{\sigma \in \mathfrak{S}_n \mid \pi(a) < \pi(b) \Rightarrow \sigma(a) < \sigma(b), \text{ for } a, b \in [\![n]\!]\}$$

is the set of linear extensions of π on $[\![n]\!]$.

Consistency assumption

Marginal operator on $A \in \mathcal{P}(\llbracket n \rrbracket)$:

$$M_A : L(\mathfrak{S}_n) \to L(\mathfrak{R}_A)$$
$$M_A f(\pi) = \sum_{\sigma \in \mathfrak{S}_n(\pi)} f(\sigma) \qquad \text{for all } \pi \in \mathfrak{R}_A.$$

Example

For $n = 3$, $f \in L(\mathfrak{S}_3)$,

$$M_{\{1,3\}} f(1 \prec 3) = f(1 \prec 3 \prec 2) + f(1 \prec 2 \prec 3) + f(2 \prec 1 \prec 3).$$

For a probability distribution p on $\mathfrak{S}_n$ and $A = \{a_1, \ldots, a_k\}$,

$$M_A p(a_{i_1} \prec \cdots \prec a_{i_k}) \equiv \mathbb{P}\left[a_{i_1} \prec \cdots \prec a_{i_k} | a_1, \ldots, a_k\right].$$

Consistency assumption

Global marginal operator on $\mathcal{A}$:

$$\begin{aligned} M_{\mathcal{A}} : L(\mathfrak{S}_n) &\to \bigoplus_{A \in \mathcal{A}} L(\mathfrak{R}_A) \\ f &\mapsto (M_A f)_{A \in \mathcal{A}}. \end{aligned}$$

Assumption $(*)$ $\Leftrightarrow$ there exists p probability distribution on $\mathfrak{S}_n$ such that

$$M_{\mathcal{A}} p = (P_A)_{A \in \mathcal{A}}.$$

Space for data analysis:

$$\mathbb{M}_{\mathcal{A}} = \{(f_A)_{A \in \mathcal{A}} \in \bigoplus_{A \in \mathcal{A}} L(\mathfrak{R}_A) \mid (f_A)_{A \in \mathcal{A}} \text{ satisfies } (*)\} = M_{\mathcal{A}}(L(\mathfrak{S}_n)).$$

Our purpose

Define *meaningful* approximation procedures in the space $\mathbb{M}_{\mathcal{A}}$, for any observation design $\mathcal{A}$.

Outline

Group-based harmonic analysis on $\mathfrak{S}_n$ - Interpretation

For $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$, define

$$\mathrm{Part}_\lambda(\llbracket n \rrbracket) = \{(A_1, \dots, A_r) \text{ ordered partition of } \llbracket n \rrbracket \mid |A_i| = \lambda_i\}$$

equipped with the action of $\mathfrak{S}_n$:

$$\sigma \cdot (A_1, \dots, A_r) = (\sigma(A_1), \dots, \sigma(A_r)).$$

Harmonic analysis on $\mathfrak{S}_n$ - Interpretation

Let p be a probability distribution on $\mathfrak{S}_n$ and Σ be a random permutation of law p. If an event $\mathcal{E}$ corresponds to a subset $S \subset \mathfrak{S}_n$, we define

$$\mathbb{P}[\mathcal{E}] = \sum_{\sigma \in S} p(\sigma).$$

For $\lambda \vdash n$ and $\mathcal{B}_0 \in \mathrm{Part}_\lambda(\llbracket n \rrbracket)$, the λ-marginal in $\mathcal{B}_0$ of p is the probability distribution on $\mathrm{Part}_\lambda(\llbracket n \rrbracket)$:

$$\left(\mathbb{P}\left[\Sigma \cdot \mathcal{B}_0 = \mathcal{B}\right]\right)_{\mathcal{B} \in \mathrm{Part}_\lambda(\llbracket n \rrbracket)}$$

Harmonic analysis on $\mathfrak{S}_n$ - Interpretation

Example: $\lambda = (n-1, 1)$

$$\mathrm{Part}_{(n-1,1)}([\![n]\!]) = \{([\![n]\!] \setminus \{i\}, \{i\}) \mid i \in [\![n]\!]\} \cong \{i \in [\![n]\!]\}$$

$(n-1,1)$-marginals are the probability distributions

$$(\mathbb{P}[\Sigma(i) = j])_{j \in [\![n]\!]}$$

i.e. the laws of the random variables $\Sigma(i)$, for $i \in [\![n]\!]$.

Harmonic analysis on $\mathfrak{S}_n$ - Interpretation

Example: $\lambda = (n-2, 2)$

$(n-2,2)$-marginals are the probability distributions

$$(\mathbb{P}[\Sigma(\{i_1, i_2\}) = \{j_1, j_2\}])_{1\leq j_1<j_2\leq n}$$

i.e. the laws of the random variables $\{\Sigma(i_1), \Sigma(i_2)\}$, for $1 \leq i_1 < i_2 \leq n$.

Example: $\lambda = (n-2, 1, 1)$

$(n-2,1,1)$-marginals are the probability distributions

$$(\mathbb{P}[\Sigma((i_1, i_2)) = (j_1, j_2)])_{1\leq j_1<j_2\leq n}$$

i.e. the laws of the random variables $(\Sigma(i_1), \Sigma(i_2))$, for $1 \leq i_1 < i_2 \leq n$.

Information localization

- $\mathfrak{S}_n$-based harmonic analysis localizes "absolute rank information":

$$\mathbb{P}[\Sigma(a_1) = i_1, \dots, \Sigma(a_k) = i_k].$$

- Incomplete rankings analysis requires to localize "relative rank information":

$$\mathbb{P}[\Sigma(a_1) < \cdots < \Sigma(a_k)].$$

Information localization - Example

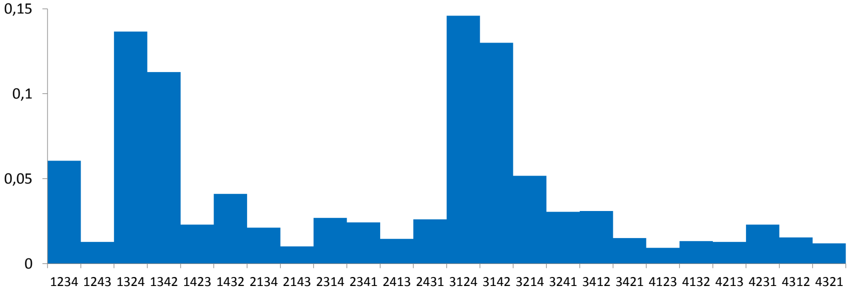


Figure: full distribution on $\mathfrak{S}_4$

Information localization - Example

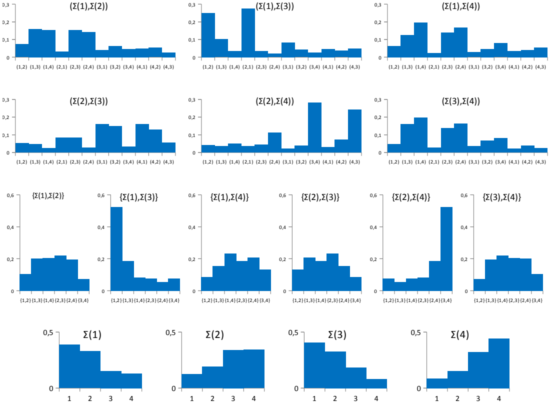


Figure: λ-marginals for $\lambda = (3,1), (2,2), (2,1,1)$

Information localization - Example

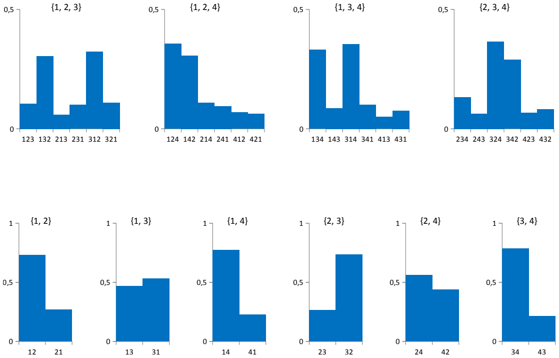


Figure: marginals on subsets $A \subset [\![4]\!]$ with $|A| \geq 2$

Outline

Decomposition of $L(\mathfrak{S}_n)$

Let $V^0 = \{f \text{ constant on } \mathfrak{S}_n\} = \mathbb{R}\psi_0$ with $\psi_0 = \mathbb{1}_{\mathfrak{S}_n}$.

Theorem

$$L(\mathfrak{S}_n) = V^0 \oplus \bigoplus_{A \in \mathcal{P}(\llbracket n \rrbracket)} W_A,$$

where W_A is a subspace of $L(\mathfrak{S}_n)$ that localizes the information specific to the marginal on A an not to the others:

1. $W_A \cap \ker M_A = \{0\}$
2. $W_A \subset \ker M_B$ *for all* $B \in \mathcal{P}(\llbracket n \rrbracket)$ *such that* $A \not\subset B$

Decomposition of a function $f \in L(\mathfrak{S}_n)$

Corollary

For $f \in L(\mathfrak{S}_n)$, denote by

$$f = \frac{\sum_{\sigma \in \mathfrak{S}_n} f(\sigma)}{n!} + \sum_{A \in \mathcal{P}([\![n]\!])} f_A$$

its associated decomposition. Then for all $B \in \mathcal{P}([\![n]\!])$,

$$M_B f = \frac{\sum_{\sigma \in \mathfrak{S}_n} f(\sigma)}{|B|!} + \sum_{A \in \mathcal{P}(B)} M_B f_A.$$

Dimension

Theorem

For any $A \in \mathcal{P}(\llbracket n \rrbracket)$,

$$\dim W_A = d_{|A|}$$

where d_k *is the number of derangements (fixed-point free permutations) on* k *elements:*

k	1	2	3	4	5	...
d_k	0	1	2	9	44	...

with $\quad d_k \sim \frac{k!}{e}$

Multiresolution interpretation - Example

$$L(\mathfrak{S}_4) = V^0 \oplus \bigoplus_{k=2}^{4} \bigoplus_{|B|=k} W_A$$

level	**dim**	
4	9	$W_{\{1,2,3,4\}}$
3	8	$W_{\{1,2,3\}} \oplus W_{\{1,2,4\}} \oplus W_{\{1,3,4\}} \oplus W_{\{2,3,4\}}$
2	6	$W_{\{1,2\}} \oplus W_{\{1,3\}} \oplus W_{\{1,4\}} \oplus W_{\{2,3\}} \oplus W_{\{2,4\}} \oplus W_{\{3,4\}}$
0	1	V^0

Wavelet framework

We exhibit an explicit "wavelet basis" Ψ of $L(\mathfrak{S}_n)$ of the form

$$\Psi = \{\psi_0\} \cup \bigcup_{A \in \mathcal{P}(\llbracket n \rrbracket)} \Psi_A$$

where Ψ_B is a basis of W_B for all $B \in \mathcal{P}(\llbracket n \rrbracket)$.

For any observation design $\mathcal{A} \subset \mathcal{P}(\llbracket n \rrbracket)$,

$$M_{\mathcal{A}}(\Psi) = \{M_{\mathcal{A}}(\psi_0)\} \cup \bigcup_{B \in \bigcup_{A \in \mathcal{A}} \mathcal{P}(A)} M_{\mathcal{A}}(\Psi_B)$$

is a basis of $\mathbb{M}_{\mathcal{A}} = M_{\mathcal{A}}(L(\mathfrak{S}_n))$.

Wavelet framework - Example

$n = 4$ and $\mathcal{A} = \{\{1,3\},\{2,4\},\{3,4\},\{1,2,3\},\{1,3,4\}\}$.

$$W_{\{1,2,3,4\}}$$

$$\mathbf{W}_{\{1,2,3\}} \oplus W_{\{1,2,4\}} \oplus \mathbf{W}_{\{1,3,4\}} \oplus W_{\{2,3,4\}}$$

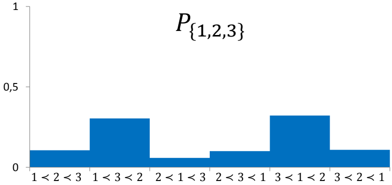
$$\mathbf{W}_{\{1,2\}} \oplus \mathbf{W}_{\{1,3\}} \oplus \mathbf{W}_{\{1,4\}} \oplus \mathbf{W}_{\{2,3\}} \oplus \mathbf{W}_{\{2,4\}} \oplus \mathbf{W}_{\{3,4\}}$$

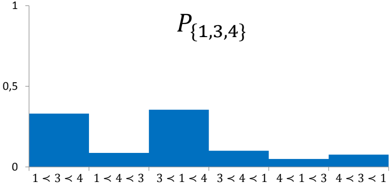
$$\mathbf{V^0}$$

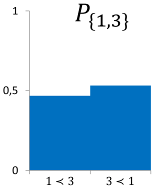
- $\mathcal{A}$ is in **orange bold**
- $\bigcup_{A\in\mathcal{A}} \mathcal{P}(A)$ is in **bold**
- $\dim \mathbb{M}_{\mathcal{A}} = 1 + \sum_{B\in\bigcup_{A\in\mathcal{A}}\mathcal{P}(A)} d_{|B|} = 1 + 6 \times 1 + 2 \times 2 = 11$

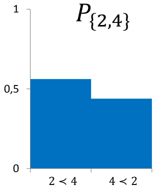
Wavelet framework - Example

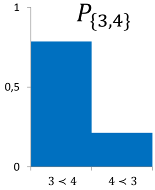
The following family of distributions...





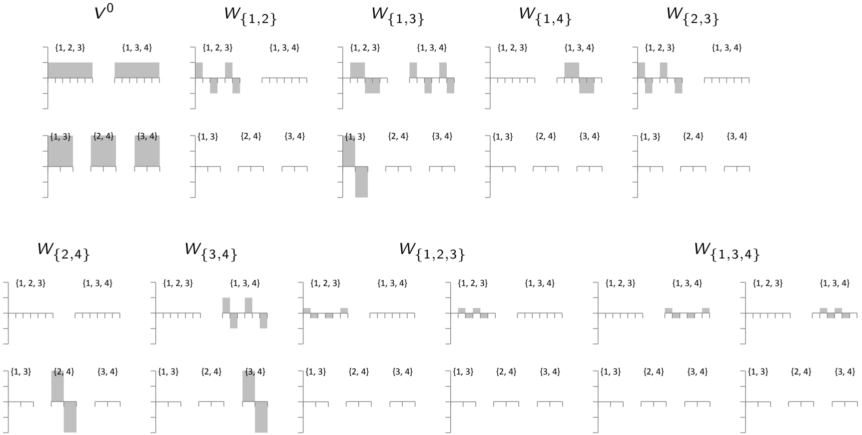






Wavelet framework - Example

... can be expanded in the wavelet basis



Outline

Motivations

Why group-based harmonic analysis on $\mathfrak{S}_n$ is not adapted

Our results

The mathematical construction

Ongoing research

Injective words

An injective word over $[\![n]\!]$ is an expression of the form $\omega_1 \dots \omega_k$ where the ω_i's are distinct elements of $[\![n]\!]$.

- $c(\omega) := \{\omega_1, \dots, \omega_k\}$ is the content of ω
- $|\omega| := |c(\omega)|$ is the length of ω

We denote by $\bar{0}$ the empty word: $c(\bar{0}) = \emptyset$ and $|\bar{0}| = 0$, and define the sets

- Γ_n: set of all injective words on $[\![n]\!]$
- Γ^k: set of injective words of size k
- $\Gamma(A)$: set of injective words of content A

$$\Gamma_n = \bigsqcup_{k=0}^{n} \Gamma^k = \bigsqcup_{k=0}^{n} \bigsqcup_{|A|=k} \Gamma(A)$$

Representation as injective words

- See incomplete rankings as injective words:

$$\mathbf{a_1} \prec \mathbf{a_2} \prec \cdots \prec \mathbf{a_k} \quad \rightarrow \quad \mathbf{a_1 a_2 \dots a_k}$$

$\Rightarrow \mathfrak{R}_A \cong \Gamma(A)$ for $A \in \mathcal{P}([\![n]\!])$

- See functions in $L(\Gamma(A))$ as functions in $L(\Gamma_n)$ that are null for $\pi \in \Gamma_n \setminus \Gamma(A)$, and denote them as free linear combinations:

$$\sum_{\pi \in \Gamma_n} f(\pi)\delta_\pi \rightarrow \sum_{\pi \in \Gamma_n} f(\pi)\pi$$

- *Rq: 0 is the function that is null for every $\pi \in \Gamma_n$ while $\bar{0}$ is the Dirac function in $\bar{0}$*

Representation as injective words

The space $L(\mathfrak{S}_n)$, the marginal spaces $L(\mathfrak{R}_A)$ for $A \in \mathcal{P}(\llbracket n \rrbracket)$ and the spaces $L(\Gamma(A))$ for $|A| \leq 1$ are all embedded in $L(\Gamma_n)$:

$$L\left(\mathfrak{S}_4\right)$$

$$L\left(\mathfrak{R}_{\{1,2,3\}}\right) \oplus L\left(\mathfrak{R}_{\{1,2,4\}}\right) \oplus L\left(\mathfrak{R}_{\{1,3,4\}}\right) \oplus L\left(\mathfrak{R}_{\{2,3,4\}}\right)$$

$$L\left(\mathfrak{R}_{\{1,2\}}\right) \oplus L\left(\mathfrak{R}_{\{1,3\}}\right) \oplus L\left(\mathfrak{R}_{\{1,4\}}\right) \oplus L\left(\mathfrak{R}_{\{2,3\}}\right) \oplus L\left(\mathfrak{R}_{\{2,4\}}\right) \oplus L\left(\mathfrak{R}_{\{3,4\}}\right)$$

$$L(\Gamma(\{1\})) \oplus L(\Gamma(\{2\})) \oplus L(\Gamma(\{3\})) \oplus L(\Gamma(\{4\}))$$

$$L(\Gamma(\bar{0}))$$

Deletion operator

Definition

For $\pi \in \Gamma_n$ and $a \in c(\pi)$, denote by $\pi \setminus \{a\}$ the word obtained by deleting the letter a in the word π.
Let $\varrho_a : L(\Gamma_n) \to L(\Gamma_n)$ be the linear operator defined on a Dirac function π by

$$\varrho_a \pi = \begin{cases} \pi \setminus \{a\} & \text{if } a \in c(\pi) \\ 0 & \text{otherwise.} \end{cases}$$

For $a_1, a_2 \in [\![n]\!]$, it is obvious that $\varrho_{a_1}\varrho_{a_2} = \varrho_{a_2}\varrho_{a_1}$.
This allows to define, for $A = \{a_1, \ldots, a_k\} \subset [\![n]\!]$, $\varrho_A = \varrho_{a_1} \ldots \varrho_{a_k}$.
We set by convention $\varrho_{\emptyset} x = x$ for all $x \in L(\Gamma_n)$.

Marginal and deletion operator

For $A \in \mathcal{P}(\llbracket n \rrbracket)$ and $\pi \in \mathfrak{R}_A$,

$$\begin{aligned}\sigma \in \mathfrak{S}_n(\pi) &\Leftrightarrow \text{for all } (a,b) \in \llbracket n \rrbracket^2, \pi(a) < \pi(b) \Rightarrow \sigma(a) < \sigma(b)\\ &\Leftrightarrow \pi \text{ is a subword of } \sigma\\ &\Leftrightarrow \sigma \setminus (\llbracket n \rrbracket \setminus A) = \pi\end{aligned}$$

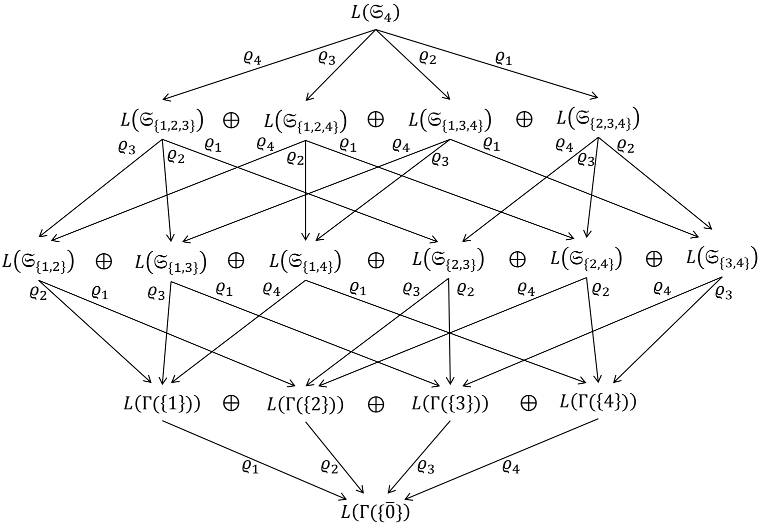
This implies that

$$M_A = \varrho_{\llbracket n \rrbracket \setminus A}$$

Projective system

The family of spaces $(L(\Gamma(A)))_{A\subset[\![n]\!]}$ equipped with the family of operators $(\varrho_{B\setminus A})_{A\subset B\subset[\![n]\!]}$ is a projective system, *i.e.* for all $A \subset B \subset C \subset [\![n]\!]$,

- $\varrho_{B\setminus A} : L(\Gamma(B)) \to L(\Gamma(A))$,
- $\varrho_{A\setminus A} x = x \qquad$ for all $x \in L(\Gamma(A))$,
- $\varrho_{B\setminus A}\varrho_{C\setminus B} = \varrho_{C\setminus A}$.


$L(\mathfrak{S}_4)$
ϱ_4 ϱ_3 ϱ_2 ϱ_1
$L(\mathfrak{S}_{\{1,2,3\}}) \oplus L(\mathfrak{S}_{\{1,2,4\}}) \oplus L(\mathfrak{S}_{\{1,3,4\}}) \oplus L(\mathfrak{S}_{\{2,3,4\}})$
ϱ_3 ϱ_2 ϱ_1 ϱ_4 ϱ_2 ϱ_1 ϱ_4 ϱ_3 ϱ_1 ϱ_4 ϱ_3 ϱ_2
$L(\mathfrak{S}_{\{1,2\}}) \oplus L(\mathfrak{S}_{\{1,3\}}) \oplus L(\mathfrak{S}_{\{1,4\}}) \oplus L(\mathfrak{S}_{\{2,3\}}) \oplus L(\mathfrak{S}_{\{2,4\}}) \oplus L(\mathfrak{S}_{\{3,4\}})$
ϱ_2 ϱ_1 ϱ_3 ϱ_1 ϱ_4 ϱ_1 ϱ_3 ϱ_2 ϱ_4 ϱ_2 ϱ_4 ϱ_3
$L(\Gamma(\{1\})) \oplus L(\Gamma(\{2\})) \oplus L(\Gamma(\{3\})) \oplus L(\Gamma(\{4\}))$
ϱ_1 ϱ_2 ϱ_3 ϱ_4
$L(\Gamma(\{\bar{0}\}))$

Spaces W_A

For $A \in \mathcal{P}(\llbracket n \rrbracket)$, the space

$$\begin{aligned} H_A &= \{f \in L(\mathfrak{R}_A) \mid M_B f = 0 \text{ for all } B \subsetneq A\} \\ &= \{f \in L(\mathfrak{R}_A) \mid M_B f = 0 \text{ for all } B \subset A \text{ with } |B| = |A| - 1\} \\ &= \{f \in L(\Gamma(A)) \mid \varrho_a f = 0 \text{ for all } a \in A\} \\ &= L(\Gamma(A)) \cap \bigcap_{a \in A} \ker \varrho_a \end{aligned}$$

localizes information of scale $|A|$ on A. We define

$$W_A = \phi_n(H_A),$$

where ϕ_n is an embedding operator $L(\Gamma_n) \to L(\mathfrak{S}_n)$.

Concatenation product

Definition

For $\pi = a_1 \dots a_r$ and $\pi' = b_1 \dots b_s$ such that $c(\pi) \cap c(\pi') = \emptyset$, define

$$\pi\pi' = a_1 \dots a_r b_1 \dots b_s.$$

It is extended as the bilinear operator $L(\Gamma_n) \times L(\Gamma_n) \to L(\Gamma_n)$ defined on Dirac functions by

$$(\pi, \pi') \mapsto \begin{cases} \pi\pi' & \text{if } c(\pi) \cap c(\pi') = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Embedding operator

For $\omega \in \Gamma_n$, let i_ω and j_ω be the two operators on $L(\Gamma_n)$ defined on the Dirac functions by

$$\mathrm{i}_\omega : \pi \mapsto \omega\pi \qquad \text{and} \qquad \mathrm{j}_\omega : \pi \mapsto \pi\omega.$$

Then define

$$\phi_n = \sum_{\substack{\omega_1,\omega_2 \in \Gamma_n \\ c(\omega_1) \sqcup c(\omega_2) \sqcup c(\pi) = [\![n]\!]}} \mathrm{i}_{\omega_1} \mathrm{j}_{\omega_2}.$$

Example

$$\phi_5(143) = 25143 + 52143 + 21435 + 51432 + 14325 + 14352.$$

Proof

Lemma

For $\omega \in \Gamma_n$ and $a \in [\![n]\!] \setminus c(\omega)$,

$$\varrho_a \mathfrak{i}_\omega = \mathfrak{i}_\omega \varrho_a \qquad \textit{and} \qquad \varrho_a \mathfrak{j}_\omega = \mathfrak{j}_\omega \varrho_a.$$

Proposition

For $A \in \mathcal{P}([\![n]\!])$, W_A localizes the information specific to A:

$$W_A \cap \ker M_A = \{0\} \qquad \textit{and} \qquad W_A \subset \bigcap_{\substack{B \in \mathcal{P}([\![n]\!]) \\ B \not\supset A}} \ker M_B.$$

Proof

Define $L_0(\mathfrak{S}_n) = \{f \in L(\mathfrak{S}_n) \mid \sum_{\sigma \in \mathfrak{S}_n} f(\sigma) = 0\}$, so that:

$$L(\mathfrak{S}_n) = V^0 \overset{\perp}{\oplus} L_0(\mathfrak{S}_n).$$

Proposition

The spaces W_A are in direct sum in $L_0(\mathfrak{S}_n)$.

Remark

The sum is not orthogonal.

Proof

Theorem (Reiner et al., 2013)

For all $A \in \mathcal{P}(\llbracket n \rrbracket)$,

$$\dim H_A = d_{|A|}.$$

The proof of the decomposition is concluded by a dimensional argument:

$$\dim \left(V^0 \oplus \bigoplus_{A \in \mathcal{P}(\llbracket n \rrbracket)} W_A \right) = 1 + \sum_{k=2}^{n} \binom{n}{k} d_k$$

$$= \sum_{k=0}^{n} \binom{n}{k} d_{n-k} = n! = \dim L(\mathfrak{S}_n).$$

Link with algebraic topology

The spaces $L(\Gamma^k)$ are canonically equipped with the operators $\partial_k : L(\Gamma^k) \to L(\Gamma^{k-1})$ defined on Dirac functions by

$$\partial_k (\pi_1 \dots \pi_k) = \sum_{m=1}^{k} (-1)^{m-1} \pi_1 \dots \pi_{m-1} \pi_{m+1} \dots \pi_k,$$

for $k \in \{1, \dots, n\}$. They are *boundary* operators: $\partial_k \partial_{k+1} = 0$. The sequence $(L(\Gamma^k), \partial_k)_{1 \le k \le n}$ is the complex of injective words:

$$L(\Gamma^n) \overset{\partial_n}{\to} L(\Gamma^{n-1}) \overset{\partial_{n-1}}{\to} \dots \overset{\partial_2}{\to} L(\Gamma^1) \overset{\partial_1}{\to} L(\Gamma^0)$$

Link with algebraic topology

Operators ∂_k are interpreted as "divergence" operators

- $\ker \partial_k$ is the k^{th} cycle space
- $\operatorname{im} \partial_{k+1}$ is the k^{th} boundary space
- $\mathcal{H}_k = \ker \partial_k / \operatorname{im} \partial_{k+1}$ is the k^{th} homology space

Spaces $\mathcal{H}_k$ have been well-studied in the literature. In particular:

- $\mathcal{H}_k = 0$ for $k \in \{1, \ldots, n-1\}$,
- $\dim \mathcal{H}_n = d_n$, where $\mathcal{H}_n := \ker \partial_n$.

Link with algebraic topology

We denote the sign of a permutation $\pi \in \Gamma^n$ by $\mathrm{sgn}(\pi)$ and define the operator $\mathrm{Sgn} : L(\mathfrak{S}_n) \to L(\mathfrak{S}_n)$ on Dirac functions by

$$\mathrm{Sgn}(\pi) = \mathrm{sgn}(\pi)\pi.$$

Proposition

The operator Sgn *is an involution between* $H_{[\![n]\!]}$ *and* $\mathcal{H}_n$*. In particular,*

$$\dim H_{[\![n]\!]} = \dim \mathcal{H}_n = d_n.$$

For $A \in \mathcal{P}([\![n]\!])$*,* H_A *is isomorphic to* $H_{\{1,\dots,|A|\}}$ *and thus to* $\mathcal{H}_{|A|}$*.*

"Wavelet basis"

- A basis $\mathcal{F}_A$ of H_A is generated via an algorithm adapted from one recently introduced (Ragnarsson et al., 2011) to compute a basis for the space $\mathcal{H}_n$.
- Then $\Psi_A = \{\phi_n(f) \mid f \in \mathcal{F}_A\}$ is a basis of W_A.
- The full "wavelet basis" is the concatenation

$$\Psi = \{\psi_0\} \cup \bigcup_{A \in \mathcal{P}(\llbracket n \rrbracket)} \Psi_A$$

"Wavelet basis"

- For $A \in \mathcal{P}(\llbracket n \rrbracket)$, let $\mathcal{D}_A$ be the embedding of derangements on A in $\mathfrak{S}_n$:

$$\mathcal{D}_A = \{\sigma \in \mathfrak{S}_n \mid \text{for all } a \in A, \sigma(a) \neq a \text{ and for all } b \notin A, \sigma(b) = b\}.$$

- A permutation is written in standard cycle form if:
  - each cycle is written with its smallest element in first position,
  - the cycles are ordered in increasing value of their smallest element.
- Define the bilinear operator $\diamond : L(\Gamma_n) \times L(\Gamma_n) \to L(\Gamma_n)$ on Dirac functions by

$$\pi \diamond \pi' = \pi\pi' - \pi'\pi.$$

“Wavelet basis”

The input is a permutation $\tau \in \mathcal{D}_A$ written in standard cycle form, and the output is a chain $f_\tau \in H_A$.

- If $\tau = \gamma_1 \ldots \gamma_r$ is a product of cycles:

$$f_{\gamma_1 \ldots \gamma_r} = f_{\gamma_1} \ldots f_{\gamma_r}$$

- If $\tau = \gamma$ is a cycle:

f_γ is given by the following recursive algorithm

"Wavelet basis" - Algorithm

Example for $A = \{1, 2, 3, 4, 5\}$ and $\gamma = (13524)$

$$
\begin{gathered}
1 \cdot 3 \cdot 5 \cdot 2 \cdot 4 \\
1 \cdot 3 \cdot 5 \cdot 2 \cdot 4 \\
1 \cdot (3 \diamond 5) \cdot 2 \cdot 4 \\
1 \cdot (3 \diamond 5) \cdot (2 \diamond 4) \\
(1 \diamond (3 \diamond 5)) \cdot (2 \diamond 4) \\
(1 \diamond (3 \diamond 5)) \diamond (2 \diamond 4)
\end{gathered}
$$

$f_{(13524)}$ is obtained by expanding the operators $\diamond$.

Outline

Objectives

- Compute the coefficients $c_\tau(f)$ of the expansion of a function f in the basis Ψ:

$$f = \sum_{\tau \in \mathfrak{S}_n} c_\tau(f) \psi_\tau$$

- Find a meaningful "regularity" condition corresponding to functions that are well approximated in this basis

$\Rightarrow$ Relies on the combinatorial structure of the construction