σ, φ, ψ and RH

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In 1983 Jean Louis Nicolas proved that the inequality

$$\frac{N_k}{\varphi(N_k)} > e^{\gamma} \log\log N_k,$$

where

- $\gamma \approx 0.577$ is the Euler Mascheroni constant,
- φ Euler totient function ,
- $N_n = \prod_{k=1}^{n} p_k$ the primorial of order n,

holds for all $k \geq 1$ if RH is true [2, Th. 2 (a)]. Conversely, if RH is false, the inequality holds for infinitely many k, and is violated for infinitely many k [2, Th. 2(b)].

We give an analogue for Dedekind ψ function defined by

$$\Psi(n) := n \prod_{p|n} (1 + \frac{1}{p}),$$

and more generally for the function

$$\psi_b(n) = n \prod_{p|n} \frac{1 - \frac{1}{p^b}}{1 - \frac{1}{p}},$$

for b real > 1, that occurs in the KMS states of Bost Connes quantum model of prime numbers [1]. The special case of b integer allows us to bound the sum of divisors function σ for $b-$free integers. Thus we can prove that Robin inequality [3] given by

$$\sigma(n) = \sum_{d|n} d < e^{\gamma} n \log\log n,$$

holds for 7-free integers n. This is joint work with Michel Planat.

References

[1] Bost J C and Connes A 1995 Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory Sel. Math. **1** 411-457.

[2] Jean-Louis Nicolas, Petites valeurs de la fonction d'Euler. J. Number Theory 17 (1983), no. 3, 375–388.

[3] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. J. Math. Pures Appl. (9) 63 (1984), 187–213.

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